Lorenz A. Gilch

Rate of Escape of Random Walks

DISSERTATION

zur Erlangung des akademischen Grades
Doktor der technischen Wissenschaften

Technische Universität Graz

Institut für mathematische Strukturtheorie

Betreuer und Erstgutachter:

Univ.-Prof. Dipl.-Ing. Dr.rer.nat. Wolfgang Woess
(TU Graz)

Zweitgutachter:

Prof. Dr. Donald I. Cartwright
(University of Sydney)

Graz, Januar 2007

Contents

Introduction **1**

1 Graphs and Random Walks **5**

1.1 Graphs 5

1.2 Random Walks on Graphs 6

I Rate of Escape of Random Walks on Free Products **11**

2 Free Products of Graphs **13**

2.1 The Free Product of Graphs 13

2.2 Random Walk on the Free Product 15

2.3 Properties of Generating Functions 17

2.4 Limit of the Random Walk 20

2.5 Convergence Criteria for Green Functions 22

3 Rate of Escape w.r.t. the Block Length **25**

3.1 Computation by Exit Times 26

3.1.1 Exit Times 26

3.1.2 Exit Points and Increments 26

3.1.3 Rate of Escape w.r.t. the Block Length 30

3.1.4 Computation of ν 35

3.2 Computation by Double Generating Functions 37

3.3 Computation by Limit Processes 40

3.4 Partial Rate of Escape w.r.t. the Block Length 45

3.5 Deviation from the Limit Path 47

3.6 Sample Computations 48
3.6.1 $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ 49
3.6.2 Free Product of Non-Cayley-Graphs 51
3.6.3 $\mathbb{Z}^2 * \mathbb{Z}^2$ and $\mathbb{Z}^2 * \mathbb{Z}/2\mathbb{Z}$ 53
3.7 Summary 58

4 Rate of Escape w.r.t. other Length Functions 59
4.1 Computation by First Exit Times 59
4.2 Computation by Double Generating Functions 61
4.3 Partial Rate of Escape w.r.t. Minimal Path Length 63
4.4 Sample Computations 65
4.4.1 Free Product of Non-Cayley-Graphs 65
4.4.2 Free Product with an Infinite Factor 66
4.5 Summary 68

II Acceleration of Lamplighter Random Walks 69

5 Lamplighter Random Walks 71
5.1 Lamplighter Graphs 71
5.2 Random Walks on Lamplighter Graphs 73

6 Acceleration of Lamplighter Random Walks 77
6.1 The Case $\delta_{\mathcal{L}} > 0$ 77
6.2 The Case $\delta_{\mathcal{L}} = 0$ 81
6.2.1 Graphs with Infinitely many Ends 81
6.2.1.1 Case $\Lambda \geq 2$ 82
6.2.1.2 Case $\Lambda = 1$ 86
6.2.2 Two-ended Graphs 87
6.3 The Case $\ell_0 = 0$ 92
6.4 Remarks 94
6.4.1 Switch-Walk-Switch Random Walk 94
6.4.2 Walk-or-Switch Random Walk 94
6.4.3 Multi-State Lamps 95

6.4.4 Markovian Distance 95
6.4.5 Greenian Distance 95
6.5 Summary 96

7 Lamplighter Tree **97**
7.1 Simple Random Walk on the Lamplighter Tree 97
7.2 Lower and Upper Bound 101
7.3 Another Lower Bound 107
7.4 Switch-Walk-Switch Random Walk 109

Acknowledgements **113**

References **115**

Introduction

Consider a transient Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ on a state space V and a suitable length function l on V representing a 'word length' with respect to the starting point of the Markov chain. For better visualization, we think of a random walk on a connected graph G with vertex set V. We are interested in whether the sequence of random variables $l(Z_n)/n$ converges almost surely to a constant, and if so, to compute this constant or to bound it. If the limit exists, it is called the *rate of escape*, or the *drift with respect to l* and it describes the speed of the random walk's escape to 'infinity'. In the first part of this work, we study this question for random walks on general free products of graphs. For transitive base graphs, we investigate in the second part of this work the relation between the drift of a lamplighter random walk and the drift of its projection onto the base graph.

We give some background material, restricting ourselves to random walks on graphs and discrete groups. On the d-dimensional grid $\mathbb{Z}^d$, where $d \geq 1$, random walks can be described by the sum of n independent and identically distributed random variables, the increments of each of the n steps. By the weak law of large numbers, the limit $\lim_{n\to\infty} \|Z_n\|/n$ exists almost surely, where $\|\cdot\|$ is the distance on the grid to the starting point of the random walk. Furthermore, this limit is the norm of the average displacement in one step and it is positive if the increments have non-zero mean vector. The first result regarding the rate of escape was presented by Kesten [19], who showed for symmetric random walks on groups that the spectral radius is 1 if $\liminf_{n\in\mathbb{N}} \mathbb{E}[l(Z_n)/n] = 0$, where $l(\cdot)$ is the minimal word length with respect to a fixed set of generators of the group. For random walks on generalized lattices, that is, graphs where $\mathbb{Z}^d$ acts with finitely many orbits, Salvatori [35] proved that the average mean displacement is zero if and only if the rate of escape is zero.

It is well-known that the rate of escape exists also for random walks on a finitely generated group equipped with a suitable metric, where the random walk arises from a probability measure on the group elements. This follows from *Kingman's subadditive ergodic theorem*; see Kingman [20], Derriennic [8] and Guivarc'h [13]. If l is the path metric of a Cayley graph, then the sequence $l(Z_n)/n$ converges almost surely to a constant and this constant is positive in the case when the group is non-amenable and the random walk is irreducible. Erschler [10], [11] investigated asymptotics of the drift of symmetric random walks on finitely generated groups. Mairesse [25] computed an explicit formula in terms of the unique solution of a system of polynomial equations for the rate of escape of random walks on the braid group.

There are many detailed results for random walks on groups acting on trees: Sawyer [36] investigated the drift of isotropic random walks on homogeneous trees describing the gene flow in a population. Sawyer and Steger [37] studied the rate of escape for anisotropic random walks in a tree. Cartwright, Kaimanovich and Woess [5] investigated the boundary of homogeneous trees and the drift of random walks on them. Nagnibeda and Woess [30, Section 5] proved that the rate of escape of random walks on trees with finitely many cone types is non-zero and give a formula for it. The drift has also been studied for isotropic random walks on affine buildings by Cartwright and Woess [7] and by Parkinson [31].

We will also deal with random walks on lamplighter graphs. If G is the Cayley graph of a group Γ, the lamplighter graph is the Cayley graph of the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \Gamma$. There are also many detailed results for random walks on wreath products: Lyons, Pemantle and Peres [23] gave a lower bound for the rate of escape of inward-biased random walks on lamplighter groups. Bertacchi [1] studied random walks on Diestel-Leader graphs and proved a strong law of large numbers regarding the rate of escape. Erschler (Dyubina) [9] proved that the drift of symmetric random walks on the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr A$, where A is a finitely generated group, is zero if and only if the random walk's projection onto A is recurrent. For symmetric random walks on iterated wreath products of the form $((F \wr \mathbb{Z}^2) \cdots \wr \mathbb{Z}^2) \wr \mathbb{Z}^2$, where F is a finite group, and $((\mathbb{Z} \wr \mathbb{Z}) \cdots \wr \mathbb{Z}) \wr \mathbb{Z}^2$, Erschler [10] proved zero drift. Revelle [34] examined the rate of escape of random walks on wreath products. He proved laws of the iterated logarithm for the inner and outer radius of escape.

There are further more general results regarding the rate of escape. The relation of the rate of escape with the entropy and growth of random walks was investigated by Kaimanovich and Vershik [15] and Kaimanovich and Woess [17]. Basing upon [15], an important link between drifts and Potential Theory was obtained by Varopoulos [38]. He proved that for symmetric finite range random walks on groups the existence of non-trivial bounded harmonic functions is equivalent to a non-zero rate of escape. In particular, it follows from these papers that for symmetric random walks on groups with finite first moment non-zero entropy is equivalent to a non-zero drift. Recently, Mathieu [27] proved zero drift for centered random walks. For non-group-invariant random walks on non-amenable graphs with infinitely many ends and hyperbolic graphs, Kaimanovich and Woess [16] proved that $\liminf_{n\to\infty} l(Z_n)/n > 0$, where $l(Z_n)$ is the graph distance at time n to the starting point. Karlsson and Ledrappier [18] linked Busemann functions with the drift of random walks on locally finite, homogeneous graphs. Blachère and Brofferio [2] introduced the Greenian metric, which arises from transition probabilities and is not induced by shortest paths, and Blachère, Haïssinsky and Mathieu [3] proved that entropy and rate of escape with respect to (w.r.t.) the Greenian metric of random walks on groups are equal.

In Part I of this work we study the question of existence of the rate of escape for random walks on free products of finitely many graphs, on which random walks are given and from which we construct in a natural way a random walk on the free product. For a restricted class of free products of finite groups, Mairesse [24] and Mairesse and Mathéus [26] have developed a specific technique for computation of the rate of escape w.r.t. the word length. These papers were the starting point for the present investigation of arbitrary free products. Our aim is to show existence of the rate of escape for various natural length functions and to compute formulas for it. The techniques that we use for rewriting probability generating functions in terms of functions on the factors of the free product were introduced independently and simultaneously by Cartwright and Soardi [6], Woess [41], Voiculescu [39] and McLaughlin [28]. After a general introduction to graphs and random walks in Chapter 1 we proceed in Part I as follows:

Having introduced the structure of free products and random walks in Chapter 2, we prove in Chapter 3 existence of the rate of escape w.r.t. the word length, and we will also compute three different, equivalent formulas for it using different techniques. Furthermore, we prove existence and give a formula for the rate of escape w.r.t. the partial word length, that is, we count only some pre-selected letters when computing the length. We will also show in this chapter that the random walk trajectory on the free product will be almost surely logarithmically close to the path described by its boundary limit. In Chapter 4, we extend the presented techniques to prove existence of the rate of escape on the free product w.r.t. length functions induced by path metrics, weights on the vertices or partial path lengths. We also compute formulas for them.

In Part II, we consider lamplighter random walks, which can be modelled by random walks on lamplighter graphs, and their drift. Starting with a transitive, connected, locally finite, weighted graph G, we think of a lamp sitting at each vertex, which can have the states 0 ('off') or 1 ('on'). Initially, all lamps are off. We think of a lamplighter walking along G and switching lamps on or off. We investigate the following model ('Switch-Walk-Switch') of a transient lamplighter random walk: the lamplighter tosses a coin for deciding whether to change the lamp at the current vertex, followed by a step to a random vertex, followed by tossing the coin once again for deciding whether to change the lamp's state at the destination vertex. This model will be generalized and described by a transient Markov chain $(Z_n)_{n\in\mathbb{N}_0}$, which represents the position of the lamplighter and the lamp configuration at time n. The weight of a path in G is the sum of the weights of its edges. If $m(\eta, x)$ is the minimal weight of a path for the lamplighter starting at some fixed vertex o of G with all lamps off to restore the configuration η and to reach some vertex x of G, then a natural length function $|(\eta, x)|$ is given by $m(\eta, x) + |\mathrm{supp}(\eta)| \cdot \delta_{\mathcal{L}}$ for some pre-selected $\delta_{\mathcal{L}} \geq 0$. Thus, $m(\eta, x)$ is the length of an optimal 'travelling salesman' tour from o to x that visits each point of $\mathrm{supp}(\eta)$. We will distinguish whether $\delta_{\mathcal{L}} > 0$ or not. Denote by X_n the projection of Z_n onto G and by $d(\cdot,\cdot)$ the metric on G induced by the weights of the edges. We are interested in the relation between the almost sure limits $\ell = \lim_{n\to\infty} |Z_n|/n$ and $\ell_0 = \lim_{n\to\infty} d(o, X_n)/n$. The number ℓ is the *rate of escape of the lamplighter random walk* and ℓ_0 is the *rate of escape of the lamplighter random walk's projection onto* G. We will prove that, under suitable assumptions on G, we have $\ell > \ell_0$, that is, the lamplighter random walk escapes w.r.t. $|\cdot|$ strictly faster to infinity than its projection onto G, on which we consider the metric $d(\cdot,\cdot)$.

It is not obvious that a lamplighter random walk is in general faster than its projection onto G: e.g., consider the Switch-Walk-Switch lamplighter random walk on $\mathbb{Z}$ with drift. Then the rate of escape of the lamplighter random walk is equal to the one of the random walk's projection onto $\mathbb{Z}$, whenever $\delta_{\mathcal{L}} = 0$. This follows from a result of Bertacchi [1].

The structure of Part II of this work is as follows: in Chapter 5, we give an introduction to lamplighter random walks on transitive graphs. In Chapter 6, we prove the acceleration of the lamplighter random walk under suitable assumptions: we will show that $\ell > \ell_0$ holds in the case $\delta_{\mathcal{L}} > 0$. If $\delta_{\mathcal{L}} = 0$ and $\ell_0 > 0$, then we prove this inequality assuming that G has at least two ends, where in the case of two-ended G we assume additionally that the edges have uniform weight 1. For $\ell_0 = 0$, the acceleration of the lamplighter random walk follows from results of Kaimanovich and Vershik [15] and of Varopoulos [38]. In Chapter 7, we consider the special case that G is a homogeneous tree: for a Walk-or-Switch lamplighter random walk (the lamplighter either walks or switches a lamp in one step), we represent the rate of escape by two formulas and compute two lower and one upper bound for the lamplighter's drift. Additionally, we compute a tighter lower bound (tighter than the one obtained by Chapter 6) for the drift of the Switch-Walk-Switch lamplighter random walk.

Chapter 1

Graphs and Random Walks

In this chapter, we give an introduction to graphs and random walks on them. Furthermore, we introduce some basic tools for later computations. As an initial remark, let us mention that we write $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

1.1 Graphs

A graph $\mathsf{G} = (V_\mathsf{G}, E_\mathsf{G})$ consists of a finite or countable set of vertices V_G and a set $E_\mathsf{G} \subseteq V_\mathsf{G}^2$, the set of oriented edges of G. Thus, $(x, y) \in E_\mathsf{G}$ means that there is an edge from x to y. Note that for sake of simplicity we exclude loops, that is, $(x, x) \notin E_\mathsf{G}$ for all $x \in V_\mathsf{G}$. Furthermore, there is at least one outgoing edge from each vertex. We select a special vertex o_G of G as the 'root'. The graph G is non-trivial if V_G has at least two vertices. The graph is called locally finite, if at each vertex $x \in V_\mathsf{G}$ there is only a finite degree number $\deg(x)$ of outgoing edges, and G has bounded vertex degree, if there is $c \in \mathbb{N}$ such that $\deg(x) \leq c$ for all $x \in V_\mathsf{G}$. An automorphism γ of G is a bijection of V_G to itself such that $(x_1, x_2) \in E_\mathsf{G}$ if and only if $(\gamma x_1, \gamma x_2) \in E_\mathsf{G}$ for all $x_1, x_2 \in V_\mathsf{G}$. The set of automorphisms is denoted by $\mathrm{AUT}(\mathsf{G})$. The graph G is transitive if for all $x, y \in V_\mathsf{G}$ there is $\gamma \in \mathrm{AUT}(\mathsf{G})$ with $\gamma x = y$. If there is a subgroup Γ_0 of $\mathrm{AUT}(\mathsf{G})$ such that these γ's with $\gamma x = y$ can be chosen from Γ_0, then we say that Γ_0 acts transitively on G. If G is transitive, then all vertices have the same degree denoted by $\deg(\mathsf{G})$.

A path from $x \in V_\mathsf{G}$ to $y \in V_\mathsf{G}$ of length n is a sequence

$$[x = x_0, x_1, \dots, x_{n-1}, x_n = y]$$

of vertices in V_G such that there is an oriented edge from x_{i-1} to x_i for all $i \in \{1, \dots, n\}$. The graph G is connected if each pair of vertices can be joined by a path. We shall assume that our graphs are root-connected, that is, for all $x \in V_{\mathsf{G}} \setminus \{o_{\mathsf{G}}\}$ there is a path from the root o_{G} to x.

The graph structure carries a distance measure denoted by $d_{\mathsf{G}}(x,y)$, where $x,y \in V_{\mathsf{G}}$, as the minimal length of all paths from x to y, if y can be reached by a path starting in x. Otherwise we set $d_{\mathsf{G}}(x,y)=\infty$. Observe that, in general, $d_{\mathsf{G}}(x,y)$ is not the length of a shortest path from y to x. For our purpose, we are mainly interested in distances from o_{G} with respect to more general distance measures and define:

Definition 1.1 (Length Function). *A length function on* G *is a function* $l: V_{\mathsf{G}} \to \mathbb{R}_{\geq}$ *with* $l(o_{\mathsf{G}})=0$.

Obviously, we consider only length functions which are adapted to the graph structure in a suitable way. E.g., with this definiton the distance measure $d_{\mathsf{G}}(o_{\mathsf{G}},\cdot)$ on G is a length function.

1.2 Random Walks on Graphs

We think of a walker starting at o_{G} and moving randomly from vertex to vertex in G. A random walk on G is given by a transition matrix or transition operator

$$P_{\mathsf{G}} = \big(p_{\mathsf{G}}(x,y)\big)_{x,y\in V_{\mathsf{G}}},$$

where $p_{\mathsf{G}}(x,y)$ describes the single step transition probability, the probability of walking in one step to y when standing at x. We write $p_{\mathsf{G}}^{(n)}(x,y)$ for the corresponding n-step transition probability, that is, the probability of walking in n steps to y when starting at x. As a general basic assumption, we suppose that $(x,y)\in E_{\mathsf{G}}$ implies $p_{\mathsf{G}}(x,y)>0$. Obviously, each random walk on any finite or countable state space can be described as a random walk on a graph, whose vertex set is the state space and the edges are induced from all positive single step transition probabilities $p_{\mathsf{G}}(x,y)$ with $x\neq y$.

The random walk has bounded range if

$$\sup\big\{d_{\mathsf{G}}(x,y)\,\big|\,x,y\in V_{\mathsf{G}}, p_{\mathsf{G}}(x,y)>0\big\}<\infty.$$

The random walk is a nearest neighbour random walk, if also $p_{\mathsf{G}}(x,y)>0$ implies $(x,y)\in E_{\mathsf{G}}$. The random walk process is described by a Markov chain denoted by a sequence of random variables $(Y_n)_{n\in\mathbb{N}_0}$, where Y_n is the random vertex at time n. Actually, we have $Y_0=o_{\mathsf{G}}$ in most cases. Otherwise, for $\mathbb{P}[\,\cdot\,|Y_0=x]$ we write for short $\mathbb{P}_x[\,\cdot\,]$, which is the probability measure on $V_{\mathsf{G}}^{\mathbb{N}_0}$ that governs the random walk starting at $x\in V_{\mathsf{G}}$. The transition operator P_{G} is irreducible if for all $x,y\in V_{\mathsf{G}}$ there is some $n\in\mathbb{N}$ such that $p_{\mathsf{G}}^{(n)}(x,y)>0$. The random walk is transient if each vertex is visited only finitely often with probability 1. Equivalently, in the transient case, almost surely each finite subset of vertices is visited only finitely often; see e.g. Woess [43, Proposition 1.17]. If G is transitive, we call also P_{G} transitive (or space-homogeneous) if there is a subgroup $\Gamma \subseteq \mathrm{AUT}(\mathsf{G})$ such that for all $x, y \in V_{\mathsf{G}}$ there is $\gamma \in \Gamma$ with $\gamma x = y$ and $p_{\mathsf{G}}(x_1, x_2) = p_{\mathsf{G}}(\gamma x_1, \gamma x_2)$ for all $x_1, x_2 \in V_{\mathsf{G}}$. In this case we call $(\mathsf{G}, P_{\mathsf{G}})$ transitive.

Considering some selected classes of graphs we are mainly interested to investigate the speed of *transient* random walks to infinity. For this purpose, we define the following characteristical escape 'speed number':

Definition 1.2 (Rate of Escape, Drift). *Let l be a length function on the graph G. If there is a constant $\mathfrak{l} \in \mathbb{R}_{\geq}$ such that*

$$\mathfrak{l} = \lim_{n\to\infty} \frac{1}{n} l(Y_n) \quad \textit{almost surely,}$$

then $\mathfrak{l}$ is called the rate of escape or drift with respect to l of $(Y_n)_{n\in\mathbb{N}_0}$.

Obviously, the rate of escape is only of interest for transient random walks. For transitive graphs equipped with a suitable metric, existence of the rate of escape with respect to the distance to o_{G} is a consequence of *Kingman's subadditive ergodic theorem* (Kingman [20]), which we formulate most suitable for our case:

Theorem 1.3. *Consider the probability space $\Omega = V_{\mathsf{G}}^{\mathbb{N}_0}$ and denote by T the time shift on Ω with $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$. If $(W_n)_{n\in\mathbb{N}_0}$ is a subadditive sequence of non-negative real-valued random variables, that is, for all $k, n \in \mathbb{N}_0$ holds $W_{k+n} \leq W_n + W_k \circ T^n$, and if W_1 is integrable, then there is a T-invariant real-valued integrable random variable W_∞ with*

$$\lim_{n\to\infty} \frac{1}{n} W_n = W_\infty \quad \textit{almost surely.}$$

Moreover, if $(\mathsf{G}, P_{\mathsf{G}})$ is transitive such that a subgroup Γ_0 of $\mathrm{AUT}(\mathsf{G})$ acts transitively on G with $p_{\mathsf{G}}(x_1, x_2) = p_{\mathsf{G}}(\gamma x_1, \gamma x_2)$ for all $\gamma \in \Gamma_0$, $x_1, x_2 \in V_{\mathsf{G}}$, and if $d(\cdot,\cdot)$ is a Γ_0-invariant metric on G with finite first moment w.r.t. P_{G}, that is, $\sum_{x\in V_{\mathsf{G}}} d(o_{\mathsf{G}}, x) p_{\mathsf{G}}(o_{\mathsf{G}}, x) < \infty$, then there is a constant ℓ such that

$$\lim_{n\to\infty} \frac{1}{n} d(o_{\mathsf{G}}, Y_n) = \ell \quad \textit{almost surely.}$$

Proof. See Derriennic [8], Guivarc'h [13] and Woess [43, Theorem 8.14]. ☐

We now introduce some basic tools for our later computations. The main concept we want to use is the technique of generating functions, which are power series with certain probabilities appearing as their coefficients.

Definition 1.4 (Green Function). *Let $x, y \in V_{\mathsf{G}}$, $z \in \mathbb{C}$. The Green function associated with the random walk on G is defined as*

$$G_{\mathsf{G}}(x,y|z) := \sum_{n\geq 0} p^{(n)}_{\mathsf{G}}(x,y)\, z^n.$$

Note the following properties of Green functions:

1. If the random walk on G is irreducible and z is real bigger than zero, then the power series $G_{\mathsf{G}}(x,y|z)$ either converge or diverge simultaneously for all $x, y \in V_{\mathsf{G}}$. See Woess [43, Lemma 1.7].
2. For $|z| < 1$, $\sum_{y\in V_{\mathsf{G}}} G_{\mathsf{G}}(x,y|z) = 1/(1-z)$ for every $x \in V_{\mathsf{G}}$.

Definition 1.5 (Generating Functions). *Let $x, y \in V_{\mathsf{G}}$, $z \in \mathbb{C}$ and denote by*

$$\begin{aligned} T_y &:= \inf\bigl\{n \in \mathbb{N}_0 \mid Y_n = y\bigr\} \in \mathbb{N}_0 \cup \{\infty\}, \\ \widehat{T}_y &:= \inf\bigl\{n \in \mathbb{N} \mid Y_n = y\bigr\} \in \mathbb{N} \cup \{\infty\} \text{ respectively,} \end{aligned}$$

the stopping time of the first visit to y, the stopping time of the first return to y respectively. We define the following generating functions associated with the random walk on G:

1. *First visit generating function:*

$$F_{\mathsf{G}}(x,y|z) := \sum_{n\geq 0} \mathbb{P}_x[T_y = n]\, z^n.$$

2. *First return generating function:*

$$U_{\mathsf{G}}(x,y|z) := \sum_{n\geq 1} \mathbb{P}_x\bigl[\widehat{T}_y = n\bigr]\, z^n.$$

3. *Last exit generating function:*

$$L_{\mathsf{G}}(x,y|z) := \sum_{n\geq 0} \mathbb{P}_x\bigl[\widehat{T}_x > n, Y_n = y\bigr]\, z^n.$$

Note that the following simple properties of these generating functions hold:

1. In particular, we have $F_{\mathsf{G}}(x,x|z) = 1$ and $F_{\mathsf{G}}(x,y|1) = \mathbb{P}_x[T_y < \infty]$.
2. We have $U_{\mathsf{G}}(x,y|z) = F_{\mathsf{G}}(x,y|z)$, if $x \neq y$.
3. It is $L_{\mathsf{G}}(x,x|z) = 1$ for all $x \in V_{\mathsf{G}}$ and all $z \in \mathbb{C}$.

The following lemma shows the relations between these generating functions:

Lemma 1.6. *Let $w, x, y \in V_{\mathsf{G}}$ and $z \in \mathbb{C}$. Then*

(i) $G_{\mathsf{G}}(x,x|z) = \frac{1}{1-U_{\mathsf{G}}(x,x|z)}$,

(ii) $G_{\mathsf{G}}(x,y|z) = F_{\mathsf{G}}(x,y|z) \cdot G_{\mathsf{G}}(y,y|z)$,

(iii) $G_{\mathsf{G}}(x,y|z) = G_{\mathsf{G}}(x,x|z) \cdot L_{\mathsf{G}}(x,y|z)$,

(iv) *If each path from w to y has to pass through x, then*
$F_{\mathsf{G}}(w,y|z) = F_{\mathsf{G}}(w,x|z) \cdot F_{\mathsf{G}}(x,y|z)$ *and*
$L_{\mathsf{G}}(w,y|z) = L_{\mathsf{G}}(w,x|z) \cdot L_{\mathsf{G}}(x,y|z)$.

Proof. For the proof of (i) and (ii), see Woess [43, Lemma 1.13]. Equation (iii) is obtained by conditioning with respect to the last visit to x before finally walking to y. Statement (iv) is obtained by conditioning with respect to the first/last visit to x, which must be visited before finally walking to y. □

Part I

Rate of Escape of Random Walks on Free Products

Chapter 2

Free Products of Graphs

In this chapter, we sketch the mathematical basics for the following chapters: we give an introduction to *free products of graphs*, lift random walks from single graphs to a random walk on their free product and exhibit the basic properties of the new random walk. Our aim is to show existence of the rate of escape for various length functions on the free product of graphs, and to give formulas for it.

2.1 The Free Product of Graphs

We now explain how to construct in a natural way a new graph from some given ones; compare with Woess [43, 9.C]. Let $\mathcal{I} := \{1, \dots, r\} \subseteq \mathbb{N}$, where $r \geq 2$. Suppose we are given a finite family of non-trivial graphs $(X_i)_{i\in\mathcal{I}} = \big((V_i, E_i)\big)_{i\in\mathcal{I}}$ with disjoint vertex sets and corresponding root vertices o_i. For $i \in \mathcal{I}$, we write $V_i^\times := V_i \setminus \{o_i\}$. The free product

$$X := (V, E) := X_1 * X_2 * \cdots * X_r$$

is constructed as follows:

- The vertices V of X are all finite 'words' with letters, also called *blocks*, from $V_i^\times$, $i \in \mathcal{I}$, such that no two successive letters come from the same V_i, that is,

$$V = \Big\{ x_1 x_2 \dots x_n \;\Big|\; n \in \mathbb{N}, x_j \in \bigcup_{i\in\mathcal{I}} V_i^\times, x_k \in V_l \Rightarrow x_{k+1} \notin V_l \Big\} \cup \Big\{ o \Big\}. \quad (2.1)$$

  The empty word is denoted by o and describes the root of X. If $u = u_1 \dots u_m \in V$ and $v = v_1 \dots v_n \in V$ with $u_m \in V_i, v_1 \notin V_i$, then uv stands for the concatenation as words. We define $uo = ou = u$ for all $u \in V$ and we regard each V_i as a subset of V, identifying each o_i with o. In the sequel, we will use the representation of vertices in V as in (2.1).

- Neighbourhood in X is given as follows: if $(x, y) \in E_i$, $i \in \mathcal{I}$, then $(ux, uy) \in E$ for all $u = u_1 \dots u_m \in V$ with $u_m \notin V_i$.

To visualize this, a copy of each X_i is attached at o, which is identified with o_i, and at each vertex x of X_i copies of all other graphs X_j, $j \neq i$, are attached, where x plays the role of o_j of the copy of X_j. This construction is then iterated. Figure 2.1 illustrates (part of) the graph structure of the free product of three graphs.

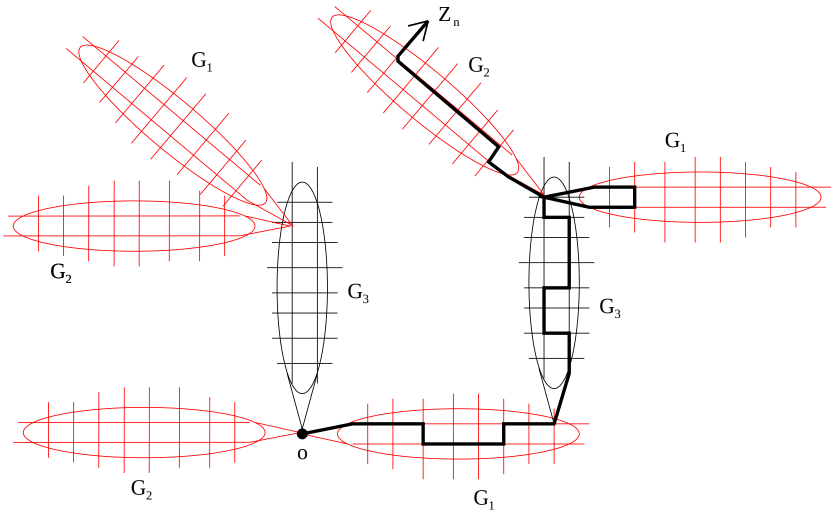


Figure 2.1: Free Product of $\mathsf{G}_1 * \mathsf{G}_2 * \mathsf{G}_3$ and Random Walk on it

We now introduce further definitions:

Definition 2.1 (Vertex Type). *Let $x = x_1 \dots x_m \in V \setminus \{o\}$ with $x_m \in V_i$, $i \in \mathcal{I}$. Then the* vertex type *of x is defined as $\tau(x) := i$. Additionally, we set $\tau(o) := 0$.*

An extension of length functions on the V_i to a length function on V is defined as follows:

Definition 2.2 (Length Function). *If we are given length functions l_i on X_i, then the associated* length function on X *is the function $l : V \to \mathbb{R}_{\geq}$ with $l(o) = 0$ and*

$$l(x_1 \dots x_m) = \sum_{j=1}^{m} l_{\tau(x_j)}(x_j) \quad \textit{for every } x_1 \dots x_m \in V \setminus \{o\}.$$

Examples 2.3:

1. Let $l_i(y) = 1$ for every $i \in \mathcal{I}$ and all $y \in V_i^\times$. Then $l(x_1 \dots x_m) = m$ for all $x = x_1 \dots x_m \in V \setminus \{o\}$. This number is called the block length of x and is denoted by $\ell(x)$.

2. Suppose we are given a metric $d_i(\cdot,\cdot)$ on the graph X_i for all $i \in \mathcal{I}$. For every vertex $x_1 \dots x_m \in V \setminus \{o\}$, the associated length function is $l(x_1 \dots x_m) = \sum_{j=1}^m d_{\tau(x_j)}(o_{\tau(x_j)}, x_j)$, which is also a metric.

3. Let $l_i(y)$ be the length of a shortest path from o_i to $y \in V_i$, $i \in \mathcal{I}$. Then the associated length function's value $l(x)$ is the minimal length of a path inside X from o to $x \in V$. It is called the minimal path length of x and is denoted by $|x|$. Observe that this is in general not the length of a shortest path from x to o, as we regard graphs with oriented edges; e.g., the case $(x, y) \in E_1$, but $(y, x) \notin E_1$ may occur.

We conclude this section with further notation. For $y \in V_i$, $i \in \mathcal{I}$, the set of successors $\mathcal{S}(y)$ of y is denoted by

$$\mathcal{S}(y) := \{y' \in V_i \mid (y, y') \in E_i\}$$

and the set of predecessors $\mathcal{P}(y)$ of y is denoted by

$$\mathcal{P}(y) := \{y' \in V_i \mid (y', y) \in E_i\}.$$

Moreover, if $k \in \mathbb{N}_0$ and $x = x_1 \dots x_m \in V \setminus \{o\}$, then the projection onto the first k blocks is defined as

$$x^{(k)} := \begin{cases} o, & \text{if } k = 0 \\ x_1 \dots x_{\min\{k,m\}}, & \text{if } k > 0 \end{cases}.$$

We also write $x^\perp := x^{(\ell(x)-1)} = x_1 \dots x_{m-1}$, if $x \neq o$. The projection onto the last block is

$$\tilde{x} := \begin{cases} x_m, & \text{if } x \neq o \\ o, & \text{if } x = o \end{cases}.$$

The cone with root $y \in V$ is defined as

$$C_y := \left\{ w \in V \;\middle|\; w^{(\ell(y))} = y \right\}.$$

2.2 Random Walk on the Free Product

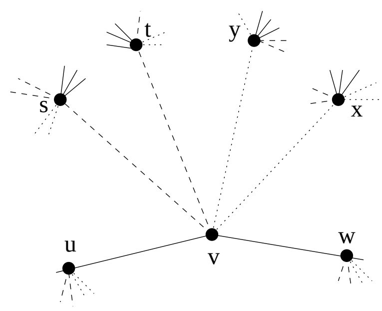
We construct a random walk on the free product X arising from random walks on the single factors X_i. Suppose we are given random walks on the graphs X_i, $i \in \mathcal{I}$, with transition matrices P_i in the sense of Section 1.2. We use the notation $p_i(x,y)$ and $p_i^{(n)}(x,y)$, $x,y \in V_i$, for the corresponding single and n-step transition probabilities. Without loss of generality we may assume that all P_i are of nearest neighbour type, as we do not assume local finiteness. Moreover, for sake of simplicity, we assume also $p_i(x,x) = 0$ for all $i \in \mathcal{I}$ and $x \in V_i$. We write $G_i(x,y|z)$, $F_i(x,y|z)$, $U_i(x,y|z)$ and $L_i(x,y|z)$ for the Green function, first visit/return and last exit generating functions associated with the random walk on X_i.

We lift P_i to a non-irreducible transition matrix $\bar{P}_i = \big(\bar{p}_i(x,y)\big)_{x,y\in V}$ on the vertex set V: if $v,w \in V_i$ and $u \in V$ with $\tau(u) \neq i$, then the single step transition probabilities are given by $\bar{p}_i(uv,uw) := p_i(v,w)$, otherwise $\bar{p}_i(x,y) := 0$. Choose $0 < \alpha_i \in \mathbb{R}$ for every $i \in \mathcal{I}$ such that $\sum_{i\in\mathcal{I}} \alpha_i = 1$. Then we obtain a new transition matrix on V given by

$$P := \sum_{i\in\mathcal{I}} \alpha_i \bar{P}_i\,;$$

compare with Woess [43, 9.C]. The random walk is a nearest neighbour random walk, since all P_i describe nearest neighbour random walks. The associated single and n-step transition probabilities are denoted by $p(x,y)$ and $p^{(n)}(x,y)$ for $x,y \in V$. The transition probabilities are sketched for a sample graph in Figure 2.2. The random walk governed by P is described by the sequence of random variables $(Z_n)_{n\in\mathbb{N}_0}$, where $Z_n \in V$ is the random vertex at time n. Initially, $Z_0 = o$ in most cases. Furthermore, denote by $\mathbb{P}_x$ the probability measure on $V^{\mathbb{N}_0}$ that governs the random walk starting at $x \in V$. We omit the index X in the notation of the corresponding generating functions, that is, we write $G(x,y|z) = G_X(x,y|z)$, $F(x,y|z) = F_X(x,y|z)$, $U(x,y|z) = U_X(x,y|z)$ and $L(x,y|z) = L_X(x,y|z)$, where $x,y \in V$, $z \in \mathbb{C}$.

Note that we make the **basic assumption** that $G(o,o|z)$ has radius of convergence *bigger than* 1. This implies *transience* of our random walk on X.



Vertex types: $\tau(u) = \tau(v) = \tau(w) = 1$,
$\tau(s) = \tau(t) = 2$, $\tau(x) = \tau(y) = 3$.

Edge	Transition probabilities
(v,w)	$\alpha_1\, p_1(v,w)$
(v,u)	$\alpha_1\, p_1(v,u)$
(v,s)	$\alpha_2\, p_2(v,s)$
(v,t)	$\alpha_2\, p_2(v,t)$
(v,x)	$\alpha_3\, p_3(v,x)$
(v,y)	$\alpha_3\, p_3(v,y)$

Figure 2.2: Transition probabilities

Thus, we may exclude the case $r = 2 = \operatorname{card} V_1 = \operatorname{card} V_2$ ($\operatorname{card} M$ is the cardinality of a set M), which reduces the free product to a line. Some criteria for this convergence property are given in Section 2.5.

2.3 Properties of Generating Functions

In this section, we present some important properties of the generating functions associated with the random walk on X. We show the correspondence between the first visit/last exit generating functions of the random walks on X and X_i. Therefore define for $i \in \mathcal{I}$ and $z \in \mathbb{C}$

$$\bar{H}_i(z) := \sum_{n=2}^{\infty} \mathbb{P}_o[T_o = n, Z_1 \notin V_i]\, z^n \quad \text{and} \quad \xi_i(z) := \frac{\alpha_i z}{1 - \bar{H}_i(z)};$$

see Woess [43, 9.C]. Note that $\bar{H}_i(1)$ is the probability of starting at $x \in V_i$ and ever returning to x without visiting any neighbour in $\mathcal{S}(x)$ before reaching x. The number $\xi_i(1)$ is the probability of starting at $x \in V_i$ and ever visiting any neighbour in $\mathcal{S}(x)$. Observe that for real $z > 0$ the functions $\bar{H}_i(z)$ and $\xi_i(z)$ are strictly increasing inside their radii of convergence.

Lemma 2.4. *Let R be the radius of convergence of $G(o,o|z)$ and for $i \in \mathcal{I}$ let R_i be the radius of convergence of $G_i(o_i,o_i|z)$. If $0 \leq z < R$, then $0 \leq \xi_i(z) < R_i$.*

Proof. See Woess [43, Proposition 9.18]; the proof does not require irreducibility. □

Note that the last lemma implies that $\xi_i(z)$, and thus $\bar{H}_i(z)$ have radii of convergence bigger than 1 for all $i \in \mathcal{I}$, as we assume $R > 1$. We can now state the following essential equation:

Proposition 2.5. *Let $i \in \mathcal{I}$, $x, y \in V_i$ and $z \in \mathbb{C}$. Then*

$$F(x,y|z) = F_i\big(x,y|\xi_i(z)\big).$$

Proof. See Woess [43, Proposition 9.18 (c)]; the proof does not require irreducibility. □

By decomposing w.r.t. the first step we can write:

$$\bar{H}_i(z) = \sum_{j \in \mathcal{I}\setminus\{i\}} \sum_{s \in \mathcal{S}(o_j)} \Big(\alpha_j \cdot p_j(o_j,s) \cdot z \cdot F_j\big(s,o_j|\xi_j(z)\big)\Big). \tag{2.2}$$

Obviously, it is $\bar{H}_i(1) \leq 1 - \alpha_i < 1$ for all $i \in \mathcal{I}$. By continuity of $\bar{H}_i(z)$, the function $1/(1-\bar{H}_i(z))$ also has radius of convergence bigger than 1.

Lemma 2.6. $\xi_i := \xi_i(1) < 1$ *for all* $i \in \mathcal{I}$.

Proof. Let $H_i(z) := U(o,o|z) - \bar{H}_i(z)$. By transience, we have

$$U(o,o|1) = \sum_{i\in\mathcal{I}} H_i(1) < 1.$$

Furthermore,

$$H_i(1) = \alpha_i \sum_{s\in\mathcal{S}(o_i)} p_i(o_i,s)\underbrace{F(s,o|1)}_{\leq 1} \leq \alpha_i.$$

Hence,

$$\xi_i = \frac{\alpha_i}{1-\sum_{j\in\mathcal{I}\setminus\{i\}} H_j(1)} \leq \frac{\alpha_i}{1-\sum_{j\in\mathcal{I}\setminus\{i\}} \alpha_j} = \frac{\alpha_i}{1-(1-\alpha_i)} = 1.$$

Observe that if $H_i(1) < \alpha_i$ for some $i \in \mathcal{I}$, then $\xi_j < 1$ for all $j \in \mathcal{I}\setminus\{i\}$. Assume $H_i(1) = \alpha_i$ for some $i \in \mathcal{I}$. Then $\bar{H}_i(1) = U(o,o|1) - H_i(1) < 1 - \alpha_i$ and thus there is $j \in \mathcal{I}$, $j \neq i$, such that $H_j(1) < \alpha_j$. Thus $\xi_i < 1$. By

$$H_i(1) = \alpha_i \sum_{s\in\mathcal{S}(o_i)} p_i(o_i,s)F(s,o|1) = \alpha_i$$

follows $F(s,o|1) = 1$ for all $s \in \mathcal{S}(o_i)$. But now we obtain the contradiction

$$1 = F(s,o|1) = F_i(s,o_i|\xi_i) < 1,$$

as $\xi_i < 1$ and $F_i(s,o_i|z)$ is strictly increasing for real $z > 0$ and $F_i(s,o_i|1) \leq 1$. This finishes the proof. □

Remark: By Lemma 2.6 and continuity of the function $\xi_i(z)$, the power series $G_i\big(o_i,o_i|\xi_i(z)\big)$ and the function $1/\big(1-\xi_i(z)\big)$, $i \in \mathcal{I}$, have radii of convergence bigger than 1.

There is an analogous important relation between the last exit generating functions associated with the random walks on the factors X_i and on X:

Proposition 2.7. *Let* $i \in \mathcal{I}$, $x,y \in V_i$ *and* $z \in \mathbb{C}$. *Then*

$$L(x,y|z) = L_i\big(x,y|\xi_i(z)\big).$$

Proof. The proof works analogously to the proof of Proposition 2.5; compare with Woess [43, Proof of Prop. 9.18 (c)]. The basic idea is as follows: we look at all walks $[x = x_0, x_1, \ldots, x_n = y]$ inside X_i starting in x and reaching y without returning to x during this walk. Writing $V_i^\perp := \{x \in V \mid x^{(1)} \notin V_i^\times\}$, we allow at each vertex x_k, $1 \leq k \leq n$, detours into the subgraph

$$X_i^\perp := \big(V_i^\perp, E \cap (V_i^\perp \times V_i^\perp)\big),$$

which is attached at x_k, with return to x_k.

Define the stopping time s:

$$s(0) = 0, \quad s(k) = \min\bigl\{n > s(k-1) \mid (Z_{n-1}, Z_n) \in E_i\bigr\}.$$

Thus, we have $Z_{s(k)-1} = Z_{s(k-1)}$, if $s(k) < \infty$. Now, $\mathbb{P}_u[Z_n = u, s(1) > n]$ is the probability of starting and returning to $u \in V_i$ after n steps without making steps inside X_i which are subject to $\alpha_i p_i(u, \cdot)$. The probabilities above are the same for all $u \in V_i$. The generating function associated with the first visit to u without making steps inside X_i equals $\bar{H}_i(z)$. Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}_u[Z_n = u, s(1) > n]\, z^n = \frac{1}{1 - \bar{H}_i(z)}.$$

Now let $n \geq 1$ and $x_1, \dots, x_n \in V_i$ and

$$w(x_1, \dots, x_n) := \mathbb{E}\Bigl(z^{s(n)} \mathbb{1}_{[s(1)=1, Z_{s(k)}=x_k\ (k=1,\dots,n)]} \Bigm| Z_0 = x\Bigr) \cdot \frac{1}{1 - \bar{H}_i(z)}.$$

We claim that

$$w(x_1, \dots, x_n) = \xi_i(z)^n \prod_{k=1}^{n} p_i(x_{k-1}, x_k).$$

For $n = 1$ we obtain:

$$\begin{aligned} w(x_1) &= z \cdot \mathbb{P}_x[Z_1 = x_1, s(1) = 1] \cdot \frac{1}{1 - \bar{H}_i(z)} \\ &= \alpha_i\, z\, p_i(x, x_1) \cdot \frac{1}{1 - \bar{H}_i(z)} = \xi_i(z)\, p_i(x, x_1). \end{aligned}$$

By induction, we conclude:

$$\begin{aligned} w(x_1, \dots, x_n) &= w(x_1, \dots, x_{n-1}) \cdot \alpha_i\, z\, p_i(x_{n-1}, x_n) \cdot \frac{1}{1 - \bar{H}_i(z)} \\ &= w(x_1, \dots, x_{n-1}) \cdot \xi_i(z) \cdot p_i(x_{n-1}, x_n) \\ &= \xi_i(z)^n \prod_{k=1}^{n} p_i(x_{k-1}, x_k). \end{aligned}$$

Let $(Y_n)_{n \in \mathbb{N}_0}$ be a random walk on X_i governed by P_i. Then we get for $x, y \in V_i$ with $x \neq y$:

$$\begin{aligned} L(x, y|z) &= \sum_{n=1}^{\infty} \sum_{x_1, \dots, x_{n-1} \in V_i \setminus \{x\}} w(x_1, \dots, x_{n-1}, y) \\ &= \sum_{n=1}^{\infty} \xi_i(z)^n\, \mathbb{P}_x\bigl[\forall j \in \{1, \dots, n-1\} : Y_j \neq x, Y_n = y\bigr] \\ &= L_i\bigl(x, y|\xi_i(z)\bigr). \end{aligned}$$

If $x = y$, then $L(x, x|z) = 1 = L_i\bigl(x, x|\xi_i(z)\bigr)$. □

We finish this section with an important corollary:

Corollary 2.8. *Let* $x = x_1 \dots x_m \in V \setminus \{o\}$. *Then*

$$L(o,x|z) = \prod_{j=1}^{m} L_{\tau(x_j)}\big(o_{\tau(x_j)}, x_j \big| \xi_{\tau(x_j)}(z)\big).$$

Proof. By the tree-like structure of the graph of the free product, the random walk starting at o has to pass through $x_1 \dots x_j$ for all $1 \leq j \leq m$. Each such point is visited for a last time, before finally walking in direction to x. Due to the structure of X, the probability of walking in n steps from $x_0 \dots x_{j-1}$ to $x_1 \dots x_j$, where $x_0 = o$, without returning to $x_0 \dots x_{j-1}$ during this walk equals the probability of walking in n steps from o to x_j without a return to o during this walk. Applying Lemmas 1.6 and Proposition 2.7, we obtain

$$L(o,x|z) = \prod_{j=1}^{m} L(x_0 \dots x_{j-1}, x_0 \dots x_j|z) = \prod_{j=1}^{m} L_{\tau(x_j)}\big(o_{\tau(x_j)}, x_j|\xi_{\tau(x_j)}(z)\big).$$

□

2.4 Limit of the Random Walk

As we have assumed transience of the random walk on X, the random walk escapes to infinity. We shall now investigate the route of escape. The information about the route's structure is the main tool for further computations. Therefore the next goal is to show that $\ell(Z_n)$ tends almost surely to infinity for $n \to \infty$. Define for $x \in V$, $i \in \mathcal{I} \setminus \{\tau(x)\}$ and $S \subseteq V_i$

$$xS := \big\{xy \mid y \in S\big\}.$$

Then we obtain:

Lemma 2.9. $\mathbb{P}_o\big[Z_n \in xV_i$ *holds for infinitely many* $n\big] = 0$.

Proof. By Lemma 1.6 and Corollary 2.8,

$$\begin{aligned}
\sum_{n\geq 0} \mathbb{P}_o[Z_n \in xV_i] &= \sum_{y \in V_i} G(o,xy|1) \\
&= \sum_{y \in V_i} G(o,o|1) \cdot L(o,x|1) \cdot L_i(o_i,y|\xi_i) \\
&= G(o,o|1) \cdot L(o,x|1) \cdot \sum_{y\in V_i} \frac{G_i(o_i,y|\xi_i)}{G_i(o_i,o_i|\xi_i)} \\
&= \frac{G(o,o|1)\cdot L(o,x|1)}{G_i(o_i,o_i|\xi_i)} \sum_{n\geq 0}\sum_{y\in V_i} p_i^{(n)}(o_i,y)\xi_i^n \\
&= \frac{G(o,o|1)\cdot L(o,x|1)}{G_i(o_i,o_i|\xi_i)} \cdot \frac{1}{1-\xi_i} < \infty.
\end{aligned}$$

The Borel-Cantelli lemma implies the proposed equation. □

Finally, we are able to specify how the random walk on X escapes to infinity. Define

$$V_\infty := \bigl\{x_1x_2\dots \bigm| \forall j \in \mathbb{N} : i_j \in \mathcal{I}, x_j \in V_{i_j}^\times, i_j \neq i_{j+1}\bigr\},$$

the set of all infinite words $x_1x_2\dots$ in which each of the letters x_j belongs to $\bigcup_{i\in\mathcal{I}} V_i^\times$ and no two consecutive letters come from the same $V_i^\times$. For $x = x_1x_2\dots \in V_\infty$, the projection $x^{(m)}$ of x onto the first m blocks is given by $x_1 \dots x_m$. Then we have:

Proposition 2.10. *$\ell(Z_n)$ tends $\mathbb{P}_o$-a.s. to infinity if $n \to \infty$. Furthermore, there exists a V_∞-valued random variable Z_∞, such that*

$$\lim_{n\to\infty} Z_n = Z_\infty \quad \mathbb{P}_o - a.s.,$$

with convergence in the sense that the length of the common prefix of Z_n and Z_∞ tends to infinity.

Proof. We prove by induction that for each $m \in \mathbb{N}$ there is almost surely some $n_m \in \mathbb{N}$ with $\ell(Z_{n_m}) = m$ and $\ell(Z_n) > m$ for all $n > n_m$. By Lemma 2.9, the random walk visits the vertex set $\bigcup_{i\in\mathcal{I}} V_i$ finitely often $\mathbb{P}_o$-a.s.. Therefore there is almost surely an index $n_1 \in \mathbb{N}$ such that

$$Z_{n_1} \in \bigcup_{i\in\mathcal{I}} V_i \quad \text{and} \quad Z_n \notin \bigcup_{i\in\mathcal{I}} V_i \quad \text{for all } n > n_1.$$

Thus $\ell(Z_{n_1}) = 1$ and $\ell(Z_n) > 1$ for all $n > n_1$. Assume now that $\ell(Z_{n_m}) = m$ and $\ell(Z_n) > m$ for all $n > n_m$. Again by Lemma 2.9, the random walk visits the vertex set $\bigcup_{i\in\mathcal{I}\setminus\{\kappa\}} Z_{n_m} V_i$ finitely often $\mathbb{P}_o$-a.s., where $\kappa := \tau(Z_{n_m})$. Then there is almost surely some $n_{m+1} \in \mathbb{N}$ such that

$$Z_{n_{m+1}} \in \bigcup_{i\in\mathcal{I}\setminus\{\kappa\}} Z_{n_m} V_i \quad \text{and } Z_n \notin \bigcup_{i\in\mathcal{I}\setminus\{\kappa\}} Z_{n_m} V_i \quad \text{for all } n > n_{m+1}.$$

Thus $\ell(Z_{n_{m+1}}) = m+1$ and $\ell(Z_n) > m+1$ for all $n > n_{m+1}$. This yields that $\ell(Z_n)$ tends almost surely to infinity if $n \to \infty$.
Obviously, the sequence $(Z_n)_{n\in\mathbb{N}_0}$ converges to an infinite word valued in V_∞ with $Z_\infty^{(m)} = Z_{n_m}$ for all $m \in \mathbb{N}$. □

Remark: A consequence of the last proposition is the fact that

$$\operatorname{card}\bigl\{m \in \mathbb{N} \bigm| \tau(Z_\infty^{(m)}) = i\bigr\} = \infty \quad \mathbb{P}_o - a.s.$$

for every $i \in \mathcal{I}$: consider V as the free product $V_i * (V_1 * \dots * V_{i-1} * V_{i+1} * \dots * V_r)$ and apply the last proposition.

We now want to investigate the speed of the escape of the random walk on X. For this purpose, we will consider the rate of escape w.r.t. different length functions. Let l be a length function on X. If there is a number $\ell \in \mathbb{R}_{\geq}$, $\lambda \in \mathbb{R}_{\geq}$ respectively, $\mathtt{l} \in \mathbb{R}_{\geq}$ respectively, such that

$$\begin{aligned}
\ell &= \lim_{n\to\infty} \frac{1}{n}\ell(Z_n) \quad \mathbb{P}_o - a.s.,\\
\lambda &= \lim_{n\to\infty} \frac{1}{n}|Z_n| \quad \mathbb{P}_o - a.s. \text{ respectively,}\\
\mathtt{l} &= \lim_{n\to\infty} \frac{1}{n}l(Z_n) \quad \mathbb{P}_o - a.s. \text{ respectively,}
\end{aligned}$$

then ℓ is called the rate of escape w.r.t. the block length, λ the rate of escape w.r.t. the minimal path length and $\mathtt{l}$ the rate of escape w.r.t. l.

We will show existence of these almost sure, constant limits and also give formulas for their computation: in Chapter 3 we will derive three formulas for the rate of escape w.r.t. the block length by three different techniques. In Chapter 4, two of the presented techniques are extended to compute the rate of escape w.r.t. the minimal path length, while one of these methods can be extended to arbitrary length functions, if card $V_i < \infty$ for every $i \in \mathcal{I}$.

2.5 Convergence Criteria for Green Functions

In the final section of this chapter we want to give three criteria such that the radius of convergence of the Green function of the random walk on a free product of graphs is *bigger than* 1 (convergence property). Recall that we have assumed this convergence property in our above computations.

A first class of random walks on free products fulfilling the necessary convergence property is the class of *uniformly irreducible, strongly reversible random walks*, when assuming that each X_i has bounded vertex degree. The random walk on the free product $X = (V, E)$ is called uniformly irreducible, if there are $N \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$(x,y) \in E \quad \text{implies} \quad p^{(n)}(x,y) \geq \varepsilon \ \text{ for some } n \leq N.$$

As we consider only nearest neighbour random walks on X, our random walk is uniformly irreducible, if the single step transition probabilities can be bounded from below by some $\varepsilon > 0$. The random walk on X is called strongly reversible, if there is a measure $m : V \to [a; b]$ with $0 < a, b \in \mathbb{R}$ and

$$m(x) \cdot p(x,y) = m(y) \cdot p(y,x) \quad \text{for all } x, y \in V.$$

Observe that reversible random walks need symmetric edge sets E, that is, $(x, y) \in E$ implies $(y, x) \in E$ for all $x, y \in V$, as we consider only nearest neighbour random walks on X. For instance, the random walk P on X is strongly reversible, if each X_i is transitive and each P_i is simple random walk on X_i.

The following theorem states the convergence property of uniformly irreducible, strongly reversible random walks on free products:

Theorem 2.11. *If each X_i has bounded vertex degree and if the random walk P on X is uniformly irreducible and strongly reversible, then the associated Green function $G(o,o|z)$ has radius of convergence bigger than* 1.

Proof. The claim follows from Woess [43, Theorems 10.6, 10.10]; compare also with Pittet [33] and Mohar [29]. □

The following lemma proposes a next class of free products fulfilling the convergence property. Recall the assumption $p_i(x,x)=0$ for all $x \in V_i$.

Lemma 2.12. *If $p_i^{(n)}(o_i,o_i)=0$ holds for some $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$, that is, the random walk P_i can not walk from any vertex in V_i to o_i, then $G(o,o|z)$ has radius of convergence bigger than* 1.

Proof. If $i \in \mathcal{I}$ such that $p_i^{(n)}(o_i,o_i)=0$ for all $n \in \mathbb{N}$, then each path inside X from o to o has to be performed outside of $V_i^\times$. Thus we obtain

$$G(o,o|z) = \sum_{n\geq 0} \underbrace{p^{(n)}(o,o)}_{\leq (1-\alpha_i)^n} z^n \leq \sum_{n\geq 0} \Big(\underbrace{(1-\alpha_i)}_{<1} z\Big)^n$$

and the proposed lemma follows. □

A final presented class of free products fulfilling the necessary convergence property of the Green function are *free products of Cayley graphs of finite or countable groups* (apart from $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$), where the random walk on these free products arise from probability measures on the generators of the single groups. We will discuss in detail this special case in Section 3.3.

Chapter 3

Rate of Escape w.r.t. the Block Length

In this chapter, we investigate the rate of escape w.r.t. the block length for the random walk on the free product $X = X_1 * \cdots * X_r$, that is, we want to show existence of a constant $\ell \in \mathbb{R}_{\geq}$ such that

$$\ell = \lim_{n\to\infty} \frac{1}{n}\ell(Z_n) \quad \text{almost surely,}$$

and we derive formulas for ℓ. For this purpose, we elaborate three different techniques to compute this limit. These techniques lead to formulas with rather different appearance, which at first glance do not seem to be related to each other. The plan for this chapter is as follows: using the notations of the previous chapter we investigate in Section 3.1 the escape of the random walk on X to infinity as precisely as possible. By purely probabilistic reasoning, we prove existence of ℓ and get a formula for it. In Section 3.2, we compute the proposed limit by dealing with double generating functions and an application of a theorem of Sawyer and Steger [37]. The third approach for the computation of ℓ in Section 3.3 works only when X is the free product of *Cayley graphs of groups* so that X itself becomes the Cayley graph of the free product of those groups. This approach investigates the behaviour of the limit process of $(Z_n)_{n\in\mathbb{N}_0}$ and uses the observations for the computation of ℓ. In Section 3.4, we give a formula for the partial rate of escape w.r.t. the block length, that is, how frequently different copies of X_i, where $i \in \mathcal{I}$ is fixed, are visited when walking to infinity. Section 3.5 shows that the random walk trajectory is logarithmically quite close to the path described by its limit process. Finally, Section 3.6 presents sample computations.

3.1 Computation by Exit Times

In this section, we investigate the random walk on X in detail, prove existence of ℓ and derive a formula for it. The basic idea behind the following technique was motivated by the paper of Nagnibeda and Woess [30, Section 5].

3.1.1 Exit Times

We introduce some random variables:

Definition 3.1 (Exit Time, Exit Point). *Let $k \in \mathbb{N}_0$. Then the exit time w.r.t. the block length k is*

$$\mathbf{e}_k := \min\Big\{m \in \mathbb{N} \;\Big|\; \forall n \geq m : Z_n^{(k)} \text{ constant}\Big\}.$$

In particular, $\mathbf{e}_0 = 0$. The exit point w.r.t. the block length k is $W_k := Z_{\mathbf{e}_k}$.

Thus, $\mathbf{e}_k$ is the first instant from which onwards the first k blocks remain constant, and $W_k = x$ if and only if at time $\mathbf{e}_k - 1$ the random walk is at state $x_1 \dots x_{k-1}s$ with some $s \in \mathcal{P}(x_k)$, at time $\mathbf{e}_k$ at state x, and thereafter remains in the cone C_x.

As Z_n converges almost surely to a random variable Z_∞ valued in V_∞, we have $\mathbf{e}_k \to \infty$ for $k \to \infty$ almost surely.

Definition 3.2 (Increment). *Let $k \in \mathbb{N}$. Then the k-th increment is given by $\mathbf{i}_k := \mathbf{e}_k - \mathbf{e}_{k-1}$.*

Definition 3.3 (Maximal Exit Time). *Let $n \in \mathbb{N}_0$. Then the random variable of the maximal exit time at time n is defined as*

$$\mathbf{k}(n) := \max\big\{k \in \mathbb{N}_0 \;\big|\; \mathbf{e}_k \leq n\big\}.$$

3.1.2 Exit Points and Increments

In this subsection we investigate the stochastic process $\big(\widetilde{W}_k, \mathbf{i}_k, \tau_k\big)_{k \in \mathbb{N}}$, where $\tau_k := \tau(W_k)$. We will show that this process is a positive recurrent Markov chain.

First, we introduce another generating function. Define for $i, j \in \mathcal{I}$ with $i \neq j$, $y \in V_j$ and $n \in \mathbb{N}_0$

$$k_i^{(n)}(o, y) := \mathbb{P}_o\big[\forall l \in \{0, \dots, n\} : Z_l \notin V_i^\times, Z_n = y\big]$$

and the corresponding generating function

$$K_i(o, y|z) := \sum_{n \geq 0} k_i^{(n)}(o, y)\, z^n = \sum_{n \geq 0} \bar{H}_i(z)^n \cdot L(o, y|z) = \frac{L(o, y|z)}{1 - \bar{H}_i(z)}. \qquad (3.1)$$

We start with the following proposition:

Proposition 3.4. *The stochastic process $(W_k, \mathbf{i}_k)_{k\in\mathbb{N}}$ is a Markov chain with transition probabilities*

$$\begin{aligned}
&\mathbb{P}_o\big[W_{k+1}=w_{k+1}, \mathbf{i}_{k+1}=n_{k+1} \,\big|\, W_k=w_k, \mathbf{i}_k=n_k\big]\\
=\;& \frac{1-\xi_{\tau(y)}}{1-\xi_{\tau(w_k)}} \cdot \sum_{s\in\mathcal{P}(y)} \Big[k_{\tau(w_k)}^{(n_{k+1}-1)}(o,s)\cdot p(s,y)\Big]
\end{aligned}$$

for $w_k = x_1\dots x_k \in V$, $w_{k+1}=w_k y$, where $y\in V_l^\times$ with $l\neq\tau(w_k)$, and $n_k, n_{k+1}\in\mathbb{N}$.

Proof. Define $\bar V_i = \bigcup_{j\in\mathcal{I}\setminus\{i\}} V_j^\times$. Let $w_0 := o$, $w_1 = g_1 \in \bigcup_{i\in\mathcal{I}} V_i^\times$ and $w_j = w_{j-1}g_j$ with $g_j\in \bar V_{\tau(w_{j-1})}$ for $2\leq j\leq k$.

For $j\in\{1,\dots,k\}$ the inclusion of events $[W_{j+1}=w_{j+1}]\subseteq[W_j=w_j]$ holds, as w_{j+1} determines the element w_j uniquely. Let $n_1,\dots,n_{k+1}\in\mathbb{N}$. We write for $m\in\{k,k+1\}$

$$\big[W_1^m = w_1^m, \mathbf{i}_1^m = n_1^m\big] := \big[\forall j\in\{1,\dots,m\}: W_j=w_j, \mathbf{i}_j=n_j\big].$$

This event can be described as follows: start at o, walk in n_1-1 steps to a predecessor of w_1 inside $X_{\tau(w_1)}$, then walk to w_1, then stay inside C_{w_1} and walk in n_2-1 steps to a vertex in $w_1\mathcal{P}(g_2)$, from there to w_2, and so on. With $n_1^s := \sum_{j=1}^s n_j$, we obtain more formally:

$$\begin{aligned}
&\mathbb{P}_o\big[W_1^k=w_1^k, \mathbf{i}_1^k=n_1^k\big]\\
=\;& \mathbb{P}_o\left[\begin{array}{c}\forall\lambda\in\{1,\dots,k\}\forall j\in\{1,\dots,n_\lambda-2\}: Z_{n_1^{\lambda-1}+j}\in C_{w_{\lambda-1}},\\ Z_{n_1^\lambda-1}\in w_{\lambda-1}\mathcal{P}(g_\lambda), Z_{n_1^\lambda}=w_\lambda; \forall j'\in\mathbb{N}: Z_{n_1^k+j'}\in C_{w_k}\end{array}\right]\\
=\;& \mathbb{P}_o\left[\begin{array}{c}\forall\lambda\in\{1,\dots,k\}\forall j\in\{1,\dots,n_\lambda-2\}:\\ Z_{n_1^{\lambda-1}+j}\in C_{w_{\lambda-1}}, Z_{n_1^\lambda-1}\in w_{\lambda-1}\mathcal{P}(g_\lambda), Z_{n_1^\lambda}=w_\lambda\end{array}\right]\\
&\cdot\mathbb{P}_{w_k}\big[\forall n\geq 1: Z_n\in C_{w_k}\big]\\
=\;& \mathbb{P}_o\left[\begin{array}{c}\forall\lambda\in\{1,\dots,k\}\forall j\in\{1,\dots,n_\lambda-2\}:\\ Z_{n_1^{\lambda-1}+j}\in C_{w_{\lambda-1}}, Z_{n_1^\lambda-1}\in w_{\lambda-1}\mathcal{P}(g_\lambda),\\ Z_{n_1^\lambda}=w_\lambda\end{array}\right]\cdot\big(1-\xi_{\tau(w_k)}\big).
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\mathbb{P}_o\big[W_1^{k+1}=w_1^{k+1}, \mathbf{i}_1^{k+1}=n_1^{k+1}\big]\\
=\;& \mathbb{P}_o\left[\begin{array}{c}\forall\lambda\in\{1,\dots,k+1\}\forall j\in\{1,\dots,n_\lambda-2\}: Z_{n_1^{\lambda-1}+j}\in C_{w_{\lambda-1}},\\ Z_{n_1^\lambda-1}\in w_{\lambda-1}\mathcal{P}(g_\lambda), Z_{n_1^\lambda}=w_\lambda; \forall j'\in\mathbb{N}: Z_{n_1^{k+1}+j'}\in C_{w_{k+1}}\end{array}\right]\\
=\;& \mathbb{P}_o\left[\begin{array}{c}\forall\lambda\in\{1,\dots,k\}\forall j\in\{1,\dots,n_\lambda-2\}:\\ Z_{n_1^{\lambda-1}+j}\in C_{w_{\lambda-1}}, Z_{n_1^\lambda-1}\in w_{\lambda-1}\mathcal{P}(g_\lambda), Z_{n_1^\lambda}=w_\lambda\end{array}\right]\\
&\cdot\mathbb{P}_{w_k}\left[\begin{array}{c}\forall j\in\{1,\dots,n_{k+1}-2\}: Z_j\in C_{w_k},\\ Z_{n_{k+1}-1}\in w_k\mathcal{P}(y), Z_{n_{k+1}}=w_k y\end{array}\right]\cdot\big(1-\xi_{\tau(y)}\big).
\end{aligned}$$

Thus we obtain the conditional probabilities:

$$\begin{aligned}
&\mathbb{P}_o\Big[W_{k+1}=w_{k+1},\mathbf{i}_{k+1}=n_{k+1}\;\Big|\;W_1^k=w_1^k,\mathbf{i}_1^k=n_1^k\Big]\\
=\;& \frac{\mathbb{P}_o\big[W_1^{k+1}=w_1^{k+1},\mathbf{i}_1^{k+1}=n_1^{k+1}\big]}{\mathbb{P}_o\big[W_1^k=w_1^k,\mathbf{i}_1^k=n_1^k\big]}\\
=\;& \frac{1-\xi_{\tau(y)}}{1-\xi_{\tau(w_k)}}\cdot\mathbb{P}_{w_k}\left[\begin{array}{c}\forall j\in\{1,\dots,n_{k+1}-2\}: Z_j\in C_{w_k},\\ Z_{n_{k+1}-1}\in w_k\mathcal{P}(y), Z_{n_{k+1}}=w_k y\end{array}\right]\\
=\;& \frac{1-\xi_{\tau(y)}}{1-\xi_{\tau(w_k)}}\cdot\sum_{s\in\mathcal{P}(y)}\mathbb{P}_{w_k}\left[\begin{array}{c}\forall j\in\{1,\dots,n_{k+1}-2\}: Z_j\in C_{w_k},\\ Z_{n_{k+1}-1}=w_k s, Z_{n_{k+1}}=w_{k+1}\end{array}\right]\\
=\;& \frac{1-\xi_{\tau(y)}}{1-\xi_{\tau(w_k)}}\cdot\sum_{s\in\mathcal{P}(y)}\Big[k_{\tau(w_k)}^{(n_{k+1}-1)}(o,s)\cdot p(s,y)\Big].
\end{aligned}$$

Therefore $(W_k,\mathbf{i}_k)_{k\in\mathbb{N}}$ is a Markov chain with the proposed transition probabilities. □

As we have seen in the last proposition, the transition probabilities of the stochastic process $(W_k,\mathbf{i}_k)_{k\in\mathbb{N}}$ depend in the present only on the vertex type of W_k and in the future only on $y=\tilde{w}_{k+1}$ and n_{k+1}. Hence, the stochastic process $\big(\widetilde{W}_k,\mathbf{i}_k,\tau_k\big)_{k\in\mathbb{N}}$ is also a Markov chain on the state space

$$\mathcal{A}:=\Big\{(y,n,j)\;\Big|\;j\in\mathcal{I},y\in V_j^\times,n\in\mathbb{N}\text{ such that }p_j^{(n)}(o,y)>0\Big\}$$

with transition probabilities

$$q\big((x,m,i),(y,n,j)\big)=\begin{cases}0, & \text{if } i=j\\ \frac{1-\xi_j}{1-\xi_i}\cdot\sum_{s\in\mathcal{P}(y)}\big[k_i^{(n-1)}(o,s)\cdot p(s,y)\big], & \text{if } i\neq j\end{cases}.$$

For reason of convenience, we set $q\big((x,m,i),(y,n,j)\big)=0$, if $j\in\mathcal{I}$, $y\in V_j$, but $(y,n,j)\notin\mathcal{A}$.

Remarks: The process $\big(\widetilde{W}_k,\mathbf{i}_k,\tau_k\big)_{k\in\mathbb{N}}$ is obviously irreducible as $\xi_i<1$ for all $i\in\mathcal{I}$. As τ_k is induced by $\widetilde{W}_k$, τ_k is superflous in the stochastic process $\big(\widetilde{W}_k,\mathbf{i}_k,\tau_k\big)_{k\in\mathbb{N}}$, but for sake of simplicity we do not drop it.

An invariant probability measure on $\mathcal{A}$ is a probability measure $\pi:\mathcal{A}\to[0;1]$ with

$$\sum_{\underline{a}\in\mathcal{A}}\pi(\underline{a})\,q(\underline{a},\underline{b})=\pi(\underline{b})$$

for all $\underline{b}\in\mathcal{A}$. We now want to show existence of an invariant probability measure for the stochastic process on $\mathcal{A}$. For this purpose, consider the stochastic process $(\tau_k)_{k\in\mathbb{N}}$. As the transition probabilities of $\big(\widetilde{W}_k, \mathbf{i}_k, \tau_k\big)_{k\in\mathbb{N}}$ depend in the present only on the vertex type τ_k, the sequence $(\tau_k)_{k\in\mathbb{N}}$ is also an irreducible Markov chain on the finite state space $\mathcal{I}$ with transition probabilities

$$\hat{q}(i,j) := \sum_{y\in V_j^{\times}} \sum_{n\geq 1} q\big((x,m,i),(y,n,j)\big)$$

for $i,j\in\mathcal{I}$ with $i\neq j$. It is $\hat{q}(i,i)=0$. Note that $x\in V_i$ and $m\in\mathbb{N}$ can be chosen arbitrarily such that $(x,m,i)\in\mathcal{A}$. Thus, $(\tau_k)_{k\in\mathbb{N}}$ possesses an invariant probability measure ν, that is, for every $j\in\mathcal{I}$

$$\sum_{i\in\mathcal{I}} \nu(i)\cdot\hat{q}(i,j) = \nu(j) \tag{3.2}$$

holds. We now define for $j\in\mathcal{I}$, $y\in V_j^{\times}$ and $n\in\mathbb{N}$:

$$\pi(y,n,j) := \sum_{i\in\mathcal{I}} \nu(i)\cdot q\big((x,m,i),(y,n,j)\big). \tag{3.3}$$

Lemma 3.5. *π is an invariant probability measure of the stochastic process $\big(\widetilde{W}_k, \mathbf{i}_k, \tau_k\big)_{k\in\mathbb{N}}$.*

Proof. We have to show for all $(y,n,j)\in\mathcal{A}$:

$$\sum_{(x,m,i)\in\mathcal{A}} \pi(x,m,i)\cdot q\big((x,m,i),(y,n,j)\big) = \pi(y,n,j).$$

Choose for each $i\in\mathcal{I}$ some $x_i\in V_i^{\times}$ and $m_i\in\mathbb{N}$ such that $(x_i,m_i,i)\in\mathcal{A}$. We prove the claim:

$$\begin{aligned}
& \sum_{(x,m,i)\in\mathcal{A}} \pi(x,m,i)\cdot q\big((x,m,i),(y,n,j)\big)\\
= & \sum_{(x,m,i)\in\mathcal{A}} \sum_{k\in\mathcal{I}} \nu(k)\cdot q\big((x_k,m_k,k),(x,m,i)\big)\cdot q\big((x,m,i),(y,n,j)\big)\\
= & \sum_{i\in\mathcal{I}} q\big((x_i,m_i,i),(y,n,j)\big)\cdot \underbrace{\sum_{k\in\mathcal{I}} \nu(k)\cdot \underbrace{\sum_{x\in V_i^{\times}} \sum_{m\in\mathbb{N}} q\big((x_k,m_k,k),(x,m,i)\big)}_{=\hat{q}(k,i)}}_{=\nu(i)}\\
= & \ \pi(y,n,j)\,.
\end{aligned}$$

□

In particular, $\big(\widetilde{W}_k, \mathbf{i}_k, \tau_k\big)_{k\in\mathbb{N}}$ is positive recurrent. In Subsection 3.1.4, we will present a formula for ν.

3.1.3 Rate of Escape w.r.t. the Block Length

In this section we finally derive a formula for ℓ. Our main tool is the following well-known theorem:

Theorem 3.6 (Ergodic Theorem for positive recurrent Markov chains). *Let $(Y_n)_{n\in\mathbb{N}}$ be an irreducible, positive recurrent Markov chain on the state space S. Let μ be the invariant probability measure on S and $f : S \to \mathbb{R}$ such that*

$$\sum_{x\in S} |f(x)|\, \mu(x) < \infty.$$

Then

$$\frac{1}{k}\sum_{l=1}^{k} f(Y_l) \xrightarrow{n\to\infty} \sum_{x\in S} f(x)\mu(x) \quad \mathbb{P}_\sigma - a.s.$$

for any initial distribution σ.

Proof. E.g. see Bremaud [4, Chapter 3, Theorem 4.1]. □

We consider the function

$$g : \mathcal{A} \to \mathbb{N} : (y, n, j) \mapsto n.$$

An application of Theorem 3.6 provides

$$\frac{1}{k}\sum_{l=1}^{k} g\big(\widetilde{W}_l, \mathbf{i}_l, \tau_l\big) = \frac{\mathbf{e}_k - \mathbf{e}_0}{k} = \frac{\mathbf{e}_k}{k} \quad \xrightarrow{k\to\infty} \quad \int g\, d\pi \quad \mathbb{P}_o - a.s.,$$

if $\int g\, d\pi < \infty$. Our aim in the sequel is now not only to show that $\int g\, d\pi$ is finite, but also to give a formula for this integral. For this purpose, we need the next three lemmas.

Lemma 3.7. *Let $i, j \in \mathcal{I}$ with $i \neq j$. Then*

$$\sum_{y\in V_j} K_i(o, y|z) = \frac{1}{1-\bar{H}_i(z)} \cdot \frac{1}{G_j\big(o_j, o_j|\xi_j(z)\big)} \cdot \frac{1}{1-\xi_j(z)}.$$

Proof. Applying equation (3.1) and Lemma 2.7, we get

$$\sum_{y\in V_j} K_i(o, y|z) = \sum_{y\in V_j} \frac{L(o, y|z)}{1-\bar{H}_i(z)} = \frac{1}{1-\bar{H}_i(z)} \sum_{y\in V_j} L_j\big(o_j, y|\xi_j(z)\big).$$

Applying Lemma 1.6 yields:

$$\begin{aligned}
\sum_{y\in V_j} L_j\big(o_j,y|\xi_j(z)\big) &= \frac{1}{G_j\big(o_j,o_j|\xi_j(z)\big)} \sum_{y\in V_j} G_j\big(o_j,y|\xi_j(z)\big)\\
&= \frac{1}{G_j\big(o_j,o_j|\xi_j(z)\big)} \sum_{n\geq 0} \underbrace{\Big(\sum_{y\in V_j} p_j^{(n)}(o_j,y)\Big)}_{=1} \xi_j(z)^n\\
&= \frac{1}{G_j\big(o_j,o_j|\xi_j(z)\big)} \cdot \frac{1}{1-\xi_j(z)}.
\end{aligned}$$

Thus, we have proved the proposed equation. ∎

Recall that $1/(1-\bar{H}_i(z))$ and $1/(1-\xi_i(z))$ have radii of convergence bigger than 1. Thus, the same holds for the sum $\sum_{y\in V_j} K_i(o,y|z)$.

Lemma 3.8. *Let $i,j\in\mathcal{I}$ with $i\neq j$. Then*

$$\sum_{s\in\mathcal{P}(o_j)} K_i(o,s|z)\,p_j(s,o_j) = \frac{G_j\big(o_j,o_j|\xi_j(z)\big)-1}{\big(1-\bar{H}_i(z)\big)\cdot G_j\big(o_j,o_j|\xi_j(z)\big)\cdot \xi_j(z)}.$$

Proof. The proposed equation is obtained as follows:

$$\begin{aligned}
&\sum_{s\in\mathcal{P}(o_j)} K_i(o,s|z)\,p_j(s,o_j)\\
=\;& \sum_{s\in\mathcal{P}(o_j)} \frac{L(o,s|z)}{1-\bar{H}_i(z)}\,p_j(s,o_j)\\
=\;& \frac{1}{1-\bar{H}_i(z)} \sum_{s\in\mathcal{P}(o_j)} \frac{G_j\big(o_j,s|\xi_j(z)\big)}{G_j\big(o_j,o_j|\xi_j(z)\big)}\,p_j(s,o_j)\\
=\;& \frac{1}{\big(1-\bar{H}_i(z)\big)\cdot G_j\big(o_j,o_j|\xi_j(z)\big)} \sum_{s\in\mathcal{P}(o_j)} G_j\big(o_j,s|\xi_j(z)\big)\,\xi_j(z)\,p_j(s,o_j)\,\frac{1}{\xi_j(z)}\\
=\;& \frac{G_j\big(o_j,o_j|\xi_j(z)\big)-1}{\big(1-\bar{H}_i(z)\big)\cdot G_j\big(o_j,o_j|\xi_j(z)\big)\cdot \xi_j(z)}.
\end{aligned}$$

∎

The radius of convergence of the term in the last lemma is bigger than 1, as $\xi_j(1)<1$ and the functions $\xi_j(z)$ and $1/\big(1-\bar{H}_i(z)\big)$ have radii of convergence bigger than 1.

Lemma 3.9. *Let $i,j\in\mathcal{I}$ with $i\neq j$. Then*

$$\gamma_{i,j}(z) := \sum_{n\geq 1}\sum_{y\in V_j^\times}\sum_{s\in\mathcal{P}(y)} k_i^{(n-1)}(o,s)\cdot p_j(s,y)\cdot z^n$$

has radius of convergence bigger than 1.

Proof. We prove:

$$
\begin{aligned}
&\sum_{n\geq 1}\sum_{y\in V_j^\times}\sum_{s\in\mathcal{P}(y)} k_i^{(n-1)}(o,s)\cdot p_j(s,y)\cdot z^n \\
=\;& \sum_{n\geq 1}\sum_{y\in V_j}\sum_{s\in\mathcal{P}(y)} k_i^{(n-1)}(o,s)\cdot p_j(s,y)\cdot z^n \\
&\quad -\sum_{n\geq 1}\sum_{s\in\mathcal{P}(o_j)} k_i^{(n-1)}(o,s)\cdot p_j(s,o_j)\cdot z^n \\
=\;& \sum_{n\geq 1}\sum_{y\in V_j} k_i^{(n-1)}(o,y)\, z^n - \sum_{n\geq 1}\sum_{s\in\mathcal{P}(o_j)} k_i^{(n-1)}(o,s)\cdot p_j(s,o_j)\cdot z^n \\
=\;& z\cdot \underbrace{\sum_{y\in V_j} K_i(o,y|z)}_{(*)} - z \underbrace{\sum_{s\in\mathcal{P}(o_j)} K_i(o,s|z)\cdot p_j(s,o_j)}_{(**)}\,. \qquad (3.4)
\end{aligned}
$$

Lemmas 3.7 and 3.8 show that the sums $(*)$ and $(**)$ have radii of convergence bigger than 1. Thus follows the claim of the lemma. □

Now we are able to prove the following theorem:

Theorem 3.10. *$\int g\,d\pi$ is finite.*

Proof. We rewrite the integral:

$$
\begin{aligned}
\int g\,d\pi \;&=\; \sum_{(y,n,j)\in\mathcal{A}} g(y,n,j)\,\pi(y,n,j) \\
&=\; \sum_{(y,n,j)\in\mathcal{A}} n\cdot\sum_{i\in\mathcal{I}}\nu(i)\,q\big((x,m,i),(y,n,j)\big) \\
&=\; \sum_{i\in\mathcal{I}}\nu(i)\cdot\sum_{(y,n,j)\in\mathcal{A}} n\cdot q\big((x,m,i),(y,n,j)\big) \\
&=\; \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}}\nu(i)\,\alpha_j\,\frac{1-\xi_j}{1-\xi_i}\underbrace{\sum_{n\geq 1} n\sum_{y\in V_j^\times}\sum_{s\in\mathcal{P}(y)} k_i^{(n-1)}(o,s)\cdot p_j(s,y)}_{(***)}
\end{aligned}
$$

We now interpret the sum $(***)$ as a power series evaluated at 1:

$$
\begin{aligned}
&\sum_{n\geq 1} n\sum_{y\in V_j^\times}\sum_{s\in\mathcal{P}(y)} k_i^{(n-1)}(o,s)\cdot p_j(s,y)\cdot z^{n-1} \\
=\;& \frac{\partial}{\partial z}\left[\sum_{n\geq 1}\sum_{y\in V_j^\times}\sum_{s\in\mathcal{P}(y)} k_i^{(n-1)}(o,s)\cdot p_j(s,y)\cdot z^n\right] \\
=\;& \gamma'_{i,j}(z).
\end{aligned}
$$

By Lemma 3.9, the sum $\gamma_{i,j}(z)$ has radius of convergence $R_{i,j} > 1$. Hence, the derivative of $\gamma_{i,j}(z)$ has the same radius of convergence $R_{i,j} > 1$. Thus, $\frac{\partial \gamma_{i,j}}{\partial z}(1) < \infty$ for all $i, j \in \mathcal{I}, i \neq j$, proving that $\int g\, d\pi$ is finite. $\square$

Having shown finiteness of $\int g\, d\pi$, we now exhibit a formula for this value. We need the following lemma:

Lemma 3.11. *For all $i, j \in \mathcal{I}$ with $i \neq j$,*

$$\gamma_{i,j}(z) = \frac{1}{\alpha_i} \cdot \frac{\xi_i(z)}{\xi_j(z)} \cdot \left(\frac{1}{\bigl(1 - \xi_j(z)\bigr) \cdot G_j\bigl(\xi_j(z)\bigr)} - 1 \right),$$

where $G_j\bigl(\xi_j(z)\bigr) := G_j\bigl(o_j, o_j | \xi_j(z)\bigr)$.

Proof. This follows directly from the previous technical Lemmas 3.7 and 3.8 and equation (3.4):

$$\begin{aligned}
\gamma_{i,j}(z) &= \frac{z}{\bigl(1 - \bar{H}_i(z)\bigr) \cdot G_j\bigl(\xi_j(z)\bigr)} \cdot \left(\frac{1}{1 - \xi_j(z)} - \frac{G_j\bigl(\xi_j(z)\bigr) - 1}{\xi_j(z)} \right) \\
&= \frac{1}{\alpha_i} \cdot \frac{\xi_i(z)}{\xi_j(z)} \cdot \frac{1}{G_j\bigl(\xi_j(z)\bigr)} \cdot \left(\frac{\xi_j(z)}{1 - \xi_j(z)} - G_j\bigl(\xi_j(z)\bigr) + 1 \right) \\
&= \frac{1}{\alpha_i} \cdot \frac{\xi_i(z)}{\xi_j(z)} \cdot \frac{1}{G_j\bigl(\xi_j(z)\bigr)} \cdot \left(\frac{1}{1 - \xi_j(z)} - G_j\bigl(\xi_j(z)\bigr) \right) \\
&= \frac{1}{\alpha_i} \cdot \frac{\xi_i(z)}{\xi_j(z)} \cdot \left(\frac{1}{\bigl(1 - \xi_j(z)\bigr) \cdot G_j\bigl(\xi_j(z)\bigr)} - 1 \right).
\end{aligned}$$

$\square$

We now obtain:

Corollary 3.12.

$$\Lambda := \int g\, d\pi = \sum_{\substack{i,j \in \mathcal{I}, \\ i \neq j}} \nu(i) \cdot \alpha_j \cdot \frac{1 - \xi_j}{1 - \xi_i} \cdot \gamma'_{i,j}(1).$$

$\square$

Moreover, we have proved by applying the *ergodic theorem for positive recurrent Markov chains*:

Corollary 3.13.

$$\frac{\mathbf{e}_k}{k} \quad \xrightarrow{k \to \infty} \quad \Lambda \quad \mathbb{P}_o - a.s.$$

$\square$

Now convergence of $\ell(Z_n)/n$ for $n \to \infty$ will be shown and a formula for the limit will be computed. For this purpose, we need the following observations:

1. $0 \leq \ell(Z_n) - \ell(W_{\mathbf{k}(n)}) \leq n - \mathbf{e}_{\mathbf{k}(n)} < \mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)}$, because the maximal difference of the block length between the words Z_n and $Z_{\mathbf{e}_{\mathbf{k}(n)}}$ equals $n - \mathbf{e}_{\mathbf{k}(n)}$.

2. $$\frac{\mathbf{e}_{\mathbf{k}(n)+1}}{\mathbf{e}_{\mathbf{k}(n)}} = \frac{\mathbf{e}_{\mathbf{k}(n)+1}}{\mathbf{k}(n)+1} \cdot \frac{\mathbf{k}(n)+1}{\mathbf{e}_{\mathbf{k}(n)}} \xrightarrow{k\to\infty} 1 \quad \mathbb{P}_o - a.s.$$

3. $$0 < \frac{\mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)}}{n} \leq \frac{\mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)}}{\mathbf{e}_{\mathbf{k}(n)}} = \frac{\mathbf{e}_{\mathbf{k}(n)+1}}{\mathbf{e}_{\mathbf{k}(n)}} - 1 \xrightarrow{n\to\infty} 0 \ \mathbb{P}_o - a.s. \tag{3.5}$$

Hence,
$$\frac{\ell(Z_n) - \ell(W_{\mathbf{k}(n)})}{n} \xrightarrow{n\to\infty} 0 \quad \mathbb{P}_o - a.s..$$
Furthermore, we have
$$\frac{\ell(W_{\mathbf{k}(n)})}{\mathbf{k}(n)} = \frac{\mathbf{k}(n)}{\mathbf{k}(n)} = 1.$$
As
$$0 \leq \frac{n}{\mathbf{e}_{\mathbf{k}(n)}} < \frac{\mathbf{e}_{\mathbf{k}(n)+1}}{\mathbf{e}_{\mathbf{k}(n)}} \xrightarrow{n\to\infty} 1 \quad \mathbb{P}_o - a.s.,$$
we can follow that
$$\frac{\mathbf{e}_{\mathbf{k}(n)}}{n} \xrightarrow{n\to\infty} 1 \quad \mathbb{P}_o - a.s.. \tag{3.6}$$
Now we can easily prove:

Corollary 3.14.

$$\frac{\ell(Z_n)}{n} \xrightarrow{n\to\infty} \ell = \frac{1}{\Lambda} \quad \mathbb{P}_o - a.s.$$

Proof. We prove:

$$\begin{aligned} & \frac{\ell(Z_n)}{n} \\ = & \frac{\ell(Z_n) - \ell(W_{\mathbf{k}(n)})}{n} + \frac{\ell(W_{\mathbf{k}(n)})}{n} \\ = & \frac{\ell(Z_n) - \ell(W_{\mathbf{k}(n)})}{n} + \frac{\ell(W_{\mathbf{k}(n)})}{\mathbf{k}(n)} \cdot \frac{\mathbf{k}(n)}{\mathbf{e}_{\mathbf{k}(n)}} \cdot \frac{\mathbf{e}_{\mathbf{k}(n)}}{n} \xrightarrow{n\to\infty} \frac{1}{\Lambda} \quad \mathbb{P}_o - a.s.. \end{aligned}$$

□

3.1.4 Computation of ν

Finally, we show in this subsection how to compute the invariant probability measure ν. For this purpose, it is sufficient to compute the transition probabilities $\hat{q}(i,j)$ for all $i,j \in \mathcal{I}$. With the help of equations (3.2) we construct a formula for ν.

If $r = 2$, then we have $\hat{q}(1,2) = \hat{q}(2,1) = 1$, leading to $\nu(1) = \nu(2) = 1/2$. For the other cases, we can derive a formula for ν with the help of the next lemma, which proposes a formula for $\hat{q}(i,j)$. For reason of better readability, we write $G_i(\xi_i) := G_i(o_i, o_i|\xi_i)$.

Lemma 3.15. *Let $i,j \in \mathcal{I}$ with $i \neq j$. Then*

$$\hat{q}(i,j) = \frac{\alpha_j}{\alpha_i} \cdot \frac{\xi_i}{\xi_j} \cdot \frac{1-\xi_j}{1-\xi_i} \cdot \left(\frac{1}{(1-\xi_j) \cdot G_j(\xi_j)} - 1 \right)$$

and $\hat{q}(i,i) = 0$.

Proof. By definition of W_k and $\mathbb{P}_o[Z_\infty \in V_\infty] = 1$ follows $\hat{q}(i,i) = 0$. With Lemma 3.11 we obtain:

$$\begin{aligned}
\hat{q}(i,j) &= \sum_{n\geq 1} \sum_{y \in V_j^\times} q\big((x,m,i),(y,n,j)\big) \\
&= \frac{1-\xi_j}{1-\xi_i} \cdot \sum_{n\geq 1} \sum_{y \in V_j^\times} \sum_{s \in \mathcal{P}(y)} \Big[k_i^{(n-1)}(o,s) \cdot \alpha_j \cdot p_j(s,y) \Big] \\
&= \frac{1-\xi_j}{1-\xi_i} \cdot \alpha_j \cdot \gamma_{i,j}(1) \\
&= \frac{\alpha_j}{\alpha_i} \cdot \frac{\xi_i}{\xi_j} \cdot \frac{1-\xi_j}{1-\xi_i} \cdot \Big(\frac{1}{(1-\xi_j) \cdot G_j(\xi_j)} - 1 \Big).
\end{aligned}$$

□

We now give an explicit formula for ν:

$$\nu(i) = c \cdot \frac{\alpha_i\,(1-\xi_i)}{\xi_i} \cdot \big(1 - (1-\xi_i) \cdot G_i(\xi_i)\big), \tag{3.7}$$

where $c > 0$ is chosen such that $\sum_{i \in \mathcal{I}} \nu(i) = 1$. This is indeed an invariant measure, because, writing $x(i) = 1 - (1-\xi_i)\,G_i(\xi_i)$, the invariance condition on ν is just

$$\sum_{i \in \mathcal{I} \setminus \{j\}} x(i) = \frac{x(j)}{\frac{1}{(1-\xi_j) \cdot G_j(\xi_j)} - 1} \qquad \text{for each } j \in \mathcal{I}$$

or, equivalently, that

$$\sum_{i\in\mathcal{I}} x(i) = 1.$$

The following lemma is the key to verify this equation.

Lemma 3.16. *Let $i \in \mathcal{I}$. Then*

$$\rho(i) := \mathbb{P}_o\Big[Z_\infty^{(1)} \in V_i^\times, \forall n \in \mathbb{N} : Z_n \notin \bigcup_{j\in\mathcal{I}\setminus\{i\}} V_j\Big] = \frac{1-(1-\xi_i)\,G_i(\xi_i)}{G(o,o|1)}.$$

Proof. By transience, o is visited only finitely often $\mathbb{P}_o$-a.s., that is,

$$\sum_{i\in\mathcal{I}} G(o,o|1)\,\rho(i) = 1. \tag{3.8}$$

This yields

$$\begin{aligned}
\rho(i) &= \sum_{y\in V_i^\times} \sum_{j\in\mathcal{I}\setminus\{i\}} L(o,y|1)\cdot\rho(j)\\
&= \sum_{y\in V_i^\times} L(o,y|1)\cdot\big(G(o,o|1)^{-1}-\rho(i)\big)\\
&= \big(G(o,o|1)^{-1}-\rho(i)\big)\sum_{y\in V_i^\times}\frac{G_i\big(o_i,y|\xi_i\big)}{G_i\big(\xi_i\big)}\\
&= \big(G(o,o|1)^{-1}-\rho(i)\big)\cdot\left(\frac{1}{(1-\xi_i)\cdot G_i(\xi_i)}-1\right).
\end{aligned}$$

This leads to the proposed equation. □

The formula for $\rho(i)$ together with equation (3.8) yields $\sum_{i\in\mathcal{I}} x(i) = 1$, that is, ν is indeed the invariant probability measure of $(\tau_k)_{k\in\mathbb{N}}$.

<u>Final remarks:</u>

If we know for each factor X_i, $i \in \mathcal{I}$, the first visit generating functions $F_i(s,o_i|z)$ for all $s \in \mathcal{P}(o_i)$, it is possible to compute $\xi_i(z)$ by solving a finite system of characteristic equations. This is in fact only possible when the generating functions are not too complicated or even unknown. Furthermore, one has to know the Green function $G_i(o_i,o_i|z)$ for each $i \in \mathcal{I}$. The measure ν can then be computed by (3.7), and thus also ℓ. Sample computations are presented in Section 3.6.

3.2 Computation by Double Generating Functions

In this section we compute the rate of escape ℓ for the random walk on X using double generating functions. The main tool for our computation is the following theorem:

Theorem 3.17 (Sawyer and Steger). *Let $(Y_n)_{n\in\mathbb{N}_0}$ be a sequence of real-valued random variables such that for some $\delta > 0$,*

$$\mathbb{E}\Biggl(\sum_{n\geq 0} \exp(-rY_n - sn)\Biggr) = \frac{C(r,s)}{g(r,s)} \qquad \text{for } 0 < r,s < \delta,$$

where $C(r,s)$ and $g(r,s)$ are analytic for $|r| < \delta$, $|s| < \delta$ and $C(0,0) \neq 0$. Denote by g_r and g_s the partial derivatives of g w.r.t. r and s. Then:

1. $$\frac{Y_n}{n} \xrightarrow{n\to\infty} C = \frac{g_r(0,0)}{g_s(0,0)} \quad \text{almost surely.}$$

2. *If $(Y_n)_{n\in\mathbb{N}_0}$ is a reversible Markov chain, then*

$$\frac{Y_n - nC}{\sqrt{n}} \xrightarrow{n\to\infty} N(0,\sigma^2) \quad \text{in law, where}$$

$$\sigma^2 = \frac{-\frac{\partial}{\partial r} g_r(0,0) + 2C\frac{\partial}{\partial s} g_r(0,0) - C^2 \frac{\partial}{\partial s} g_s(0,0)}{g_s(0,0)} \geq 0.$$

Proof. See Sawyer and Steger [37, Theorem 2.2]. □

We generalize the next considerations for later computations and we denote by $l(\cdot)$ a length function on X. Setting $Y_n = l(Z_n)$, $w = e^{-r}$ and $z = e^{-s}$, we apply Theorem 3.17 for the computation of ℓ and – in the next chapter – the rate of escape w.r.t. to any other length function l on X. It is sufficient to investigate the double generating function

$$\mathcal{E}(w,z) := \sum_{x\in V}\sum_{n\geq 0} p^{(n)}(o,x)\, w^{l(x)}\, z^n = \sum_{x\in V} G(o,x|z)\, w^{l(x)} \tag{3.9}$$

and to find for it a representation of the form

$$\frac{C(w,z)}{g(w,z)} \qquad \text{for } \ 1-\delta < w,z < 1 \text{ for some } \ \delta > 0$$

such that $C(w,z)$ and $g(w,z)$ are analytic for $|w-1|, |z-1| < \delta$ and $C(1,1) \neq 0$. Note that $\mathcal{E}(w,z)$ converges if $0 < w,z < 1$:

$$\mathcal{E}(w,z) = \sum_{n\geq 0}\sum_{x\in V} p^{(n)}(o,x) w^{l(x)} z^n \leq \sum_{n\geq 0} z^n < \infty\,.$$

We now introduce further double generating functions arising from last exit generating functions. Define for real $w,z>0$

$$\begin{aligned}
\mathcal{L}(w,z) &:= \sum_{x\in V} L(o,x|z)\, w^{l(x)} \\
&= 1+\sum_{n\geq 1}\sum_{x=x_1\dots x_n\in V\setminus\{o\}}\prod_{j=1}^{n} w^{l(x_j)} L_{\tau(x_j)}\bigl(o_{\tau(x_j)},x_j|\xi_{\tau(x_j)}(z)\bigr)
\end{aligned}$$

and for $i\in\mathcal{I}$

$$\begin{aligned}
\mathcal{L}_i^+(w,z) &:= \sum_{x\in V_i^\times} L_i\bigl(o_i,x|\xi_i(z)\bigr)\, w^{l(x)}, \\
\mathcal{L}_i(w,z) &:= \mathcal{L}_i^+(w,z)\cdot \\
&\quad \Biggl(1+\sum_{n\geq 2}\sum_{\substack{x_2\dots x_n\in V\setminus\{o\},\\ \tau(x_2)\neq i}}\prod_{j=2}^{n} w^{l(x_j)} L_{\tau(x_j)}\bigl(o_{\tau(x_j)},x_j|\xi_{\tau(x_j)}(z)\bigr)\Biggr).
\end{aligned}$$

Thus, we have the equation

$$\mathcal{L}(w,z)=1+\sum_{i\in\mathcal{I}}\mathcal{L}_i(w,z).$$

If $w,z\in\mathbb{R}$ with $0<w,z<1$, then convergence of $\mathcal{L}(w,z)$ follows by $\mathcal{L}(w,z)\leq\mathcal{E}(w,z)$, and thus convergence of $\mathcal{L}_i^+(w,z)$ and $\mathcal{L}_i(w,z)$ for each $i\in\mathcal{I}$. The next lemma proposes another representation of $\mathcal{L}(w,z)$:

Lemma 3.18. *Let $w,z\in\mathbb{R}$ with $0<w,z<1$. Then*

$$\mathcal{L}(w,z)=\frac{1}{1-\mathcal{L}^*(w,z)}\,,\qquad \textit{where } \mathcal{L}^*(w,z):=\sum_{i\in\mathcal{I}}\frac{\mathcal{L}_i^+(w,z)}{1+\mathcal{L}_i^+(w,z)}.$$

Proof. Let $w,z\in\mathbb{R}$ with $0<w,z<1$. First, we have

$$\mathcal{L}_i(w,z)=\mathcal{L}_i^+(w,z)\cdot\Bigl(1+\sum_{j\in\mathcal{I}\setminus\{i\}}\mathcal{L}_j(w,z)\Bigr)\qquad\text{for } i\in\mathcal{I},$$

and by convergence of $\mathcal{L}(w,z)$ follows

$$\mathcal{L}_i(w,z)=\mathcal{L}_i^+(w,z)\cdot\bigl(\mathcal{L}(w,z)-\mathcal{L}_i(w,z)\bigr)\,.$$

As $\mathcal{L}_i^+(w,z)>0$ holds, the last equation is equivalent to

$$\mathcal{L}_i(w,z)=\frac{\mathcal{L}_i^+(w,z)}{1+\mathcal{L}_i^+(w,z)}\mathcal{L}(w,z)\,.$$

Thus,

$$\mathcal{L}(w,z)=1+\sum_{i\in\mathcal{I}}\mathcal{L}_i(w,z)=1+\mathcal{L}^*(w,z)\,\mathcal{L}(w,z).$$

As $\mathcal{L}(w,z)<\infty$ and $\mathcal{L}^*(w,z)>0$, we have $1\neq\mathcal{L}^*(w,z)<\infty$. This yields the proposed equation. ∎

Corollary 3.19. *Let $w, z \in \mathbb{R}$ with $0 < w, z < 1$. Then*

$$\mathcal{E}(w,z) = \frac{G(o,o|z)}{1-\mathcal{L}^*(w,z)}.$$

Proof. Let $w, z \in \mathbb{R}$ with $0 < w, z < 1$. Applying Lemmas 1.6 and 3.18 yields the proposed equation:

$$\mathcal{E}(w,z) = \sum_{x\in V} G(o,o|z)\, L(o,x|z)\, w^{l(x)} = G(o,o|z)\, \mathcal{L}(w,z) = \frac{G(o,o|z)}{1-\mathcal{L}^*(w,z)}.$$

□

We can now conclude and compute a formula for the rate of escape ℓ. For this purpose, let now for the rest of this section be $l(\cdot)$ the block length, that is, $l(x) = \ell(x)$ for all $x \in V$. Then

$$\mathcal{L}_i^+(w,z) = w \cdot \sum_{x\in V_i^\times} L_i\big(o_i, x|\xi_i(z)\big) \quad \text{for each } i \in \mathcal{I}.$$

This yields

$$\begin{aligned}
\mathcal{L}^*(w,z) &= \sum_{i\in\mathcal{I}} \frac{w\sum_{x\in V_i^\times} L_i\big(o_i,x|\xi_i(z)\big)}{1+w\sum_{x\in V_i^\times} L_i\big(o_i,x|\xi_i(z)\big)} \\
&= \sum_{i\in\mathcal{I}} \frac{w\sum_{x\in V_i^\times} G_i\big(o_i,x|\xi_i(z)\big)}{G_i\big(o_i,o_i|\xi_i(z)\big)+w\sum_{x\in V_i^\times} G_i\big(o_i,x|\xi_i(z)\big)} \\
&= \sum_{i\in\mathcal{I}} \frac{w\Big(\frac{1}{1-\xi_i(z)} - G_i\big(o_i,o_i|\xi_i(z)\big)\Big)}{G_i\big(o_i,o_i|\xi_i(z)\big)+w\Big(\frac{1}{1-\xi_i(z)} - G_i\big(o_i,o_i|\xi_i(z)\big)\Big)} \\
&= \sum_{i\in\mathcal{I}} \frac{w\Big(\frac{1}{1-\xi_i(z)} - G_i\big(o_i,o_i|\xi_i(z)\big)\Big)}{\frac{w}{1-\xi_i(z)}+(1-w)G_i\big(o_i,o_i|\xi_i(z)\big)}. \qquad (3.10)
\end{aligned}$$

Now we define

$$C(w,z) := G(o,o|z) \quad \text{and} \quad g(w,z) := 1-\mathcal{L}^*(w,z),$$

yielding

$$\mathcal{E}(w,z) = \frac{C(w,z)}{g(w,z)} \quad \text{for } 0 < w, z < 1.$$

There is $\delta > 0$ such that $C(w,z)$ and $g(w,z)$ are analytic for all $w, z \in \mathbb{C}$ with $|w-1|$, $|z-1| < \delta$. This is due to the fact that

- $G(o,o|z)$ and $\xi_i(z)$ are continuous and have radii of convergence bigger than 1,

- the inequality $\xi_i(z) \neq 1$ holds in a neighbourhood of 1 by continuity of $\xi_i(z)$ and
- the continuous denominators in (3.10) evaluated at $(1,1)$ are non-zero, providing that these denominators are non-zero in a neighbourhood of 1.

Furthermore, $C(1,1) = G(o,o|1) \neq 0$. Hence, all required conditions for an application of Theorem 3.17 are fulfilled. The derivatives of g w.r.t. to r and s evaluated at $(0,0)$ are given by

$$\begin{aligned} g_r(0,0) &= g_w(e^0,e^0)\cdot(-e^0) = -g_w(1,1) \quad \text{and} \\ g_s(0,0) &= g_z(e^0,e^0)\cdot(-e^0) = -g_z(1,1), \end{aligned} \tag{3.11}$$

where g_w and g_z are the derivatives of g w.r.t. w and z. Thus, we can conclude:

$$\frac{\ell(Z_n)}{n} \xrightarrow{n\to\infty} \ell = \frac{g_w(1,1)}{g_z(1,1)} \quad \mathbb{P}_o - a.s..$$

Simplifications yield the following formula for ℓ:

Corollary 3.20. *Write $G_i(\xi_i) := G_i(o_i,o_i|\xi_i)$ and $G_i'(\xi_i) := G_i'(o_i,o_i|\xi_i)$, which is the derivative of $G_i(o_i,o_i|z)$ w.r.t. z evaluated at ξ_i. Then*

$$\ell = \frac{\sum_{i\in\mathcal{I}}\Big[\big(1-(1-\xi_i)\,G_i(\xi_i)\big)\cdot G_i(\xi_i)\cdot(1-\xi_i)\Big]}{\sum_{i\in\mathcal{I}}\Big[\xi_i'(1)\cdot\big(G_i(\xi_i)-(1-\xi_i)\cdot G_i'(\xi_i)\big)\Big]} > 0\,.$$

Proof. It remains to show that $\ell > 0$: for all $i \in \mathcal{I}$, it is $\xi_i < 1$, $G_i(\xi_i) > 0$ and

$$G_i(\xi_i) < \frac{1}{1-\xi_i}.$$

Thus, we have $\ell > 0$. □

3.3 Computation by Limit Processes

In this section we present a third technique for the computation of the rate of escape ℓ w.r.t. the block length for the random walk on the free product. This technique is restricted to the case of a free product of *Cayley graphs of groups*, which is then itself the Cayley graph of the free product of those groups.

Let $(\Gamma_i)_{i\in\mathcal{I}}$ be a finite family of non-trivial *finite* or *countable* groups. We assume that the groups have pairwise trivial intersections, but may be isomorphic. Denote by e_i the identity on Γ_i and write $\Gamma_i^\times := \Gamma \setminus \{e_i\}$. The elements of the free product group $\Gamma := \Gamma_1 * \cdots * \Gamma_r$ are all finite words

$$g_1 g_2 \cdots g_n$$

over the alphabet $\bigcup_{i\in\mathcal{I}}\Gamma_i^\times$, such that no two consecutive letters come from the same $\Gamma_i^\times$. The empty word is described by e, and e_i as a word in Γ is identified with e. Furthermore, we write $\Gamma^\times = \Gamma\setminus\{e\}$. Analogously, the block length $\ell(g_1\dots g_n)$ of $g_1\dots g_n\in\Gamma$ is n and $\ell(e):=0$.

We can define a *group operation* on Γ: the product of $u,v\in\Gamma$ is the concatenation of the words u and v with possible cancellations and contractions in the middle to get the representative form of the product word. For instance, if $u_1,u_2\in\Gamma_1, v_1,v_2\in\Gamma_2$, then $(u_1v_1u_2)\circ(u_2^{-1}v_2u_2)=u_1wu_2$, where $w=v_1v_2\in\Gamma_2$. For details, see e.g. the book by Lyndon and Schupp [22]. Recall that we exclude the case $r=2=\operatorname{card}\Gamma_1=\operatorname{card}\Gamma_2$. This ensures that the free product group Γ is not *amenable* (e.g. compare with Paterson [32]), yielding that each of our constructed random walks on Γ is transient and that $G(e,e|z)$ has radius of convergence bigger than 1; compare e.g. also with Woess [43, Theorem 10.10, Proposition 12.4, Corollary 12.5].

The random walk on Γ starting at e is constructed as follows: suppose we are given probability measures μ_i on $\Gamma_i^\times$ for each $i\in\mathcal{I}$ such that μ_i defines an irreducible random walk on Γ_i, that is, $p_i(x,y)=\mu_i(x^{-1}y)$ for all $x,y\in\Gamma_i$. Additionally, we set $\mu_i(e_i):=0$. Choose $\alpha_1,\dots,\alpha_r>0$ with $\sum_{i\in\mathcal{I}}\alpha_i=1$. Then we define the transition probabilities on Γ as

$$p(x,xg)=\begin{cases}\alpha_i\cdot\mu_i(g), & \text{if } g\in\Gamma_i^\times\\ 0, & \text{otherwise}\end{cases},$$

where $x\in\Gamma$. As the transition probabilities depend only on $g\in\Gamma_i$, we write $\mu(g):=p(x,xg)$ for all $g\in\bigcup_{i\in\mathcal{I}}\Gamma_i^\times$, and $\mu(g):=0$ otherwise. The n-fold convolution power of μ is denoted by $\mu^{(n)}$, and $p^{(n)}(x,xg)=\mu^{(n)}(g)$ for all $g\in\Gamma$. Observe that if X_i is the Cayley graph of Γ_i w.r.t. the set of generators $\{g\in\Gamma_i\mid\mu_i(g)>0\}$, then there is a one-to-one correspondence between the above defined random walk on Γ and the associated random walk on $X=X_1*\dots*X_r$, which is the Cayley graph of Γ w.r.t. the set of generators $\{g\in\Gamma\mid\mu(g)>0\}$. In other words, the random walk on Γ represents the random walk on X arising from the random walks on the X_i's in the sense of Section 2.2. Thus, we may use all previous results and notation, since the definitions in this section constitute a special case of the more general case of free products of graphs. In particular, the block length of $g\in\Gamma$ is the block length of the vertex in X representing g. Observe that X as a Cayley graph is *transitive*. Thus, existence of the rate of escape ℓ follows by an easy consequence of Kingman's subaddditive ergodic theorem (or as well by our previous results). We will see that we can drop irreducibility of the μ_i's, which we assumed only for sake of simplicity and tracing back the case of free products of groups to the case of general free products of graphs.

Furthermore, we denote by

$$\Gamma_\infty := \left\{ x_1 x_2 \ldots \;\middle|\; \forall j \in \mathbb{N} : i_j \in \mathcal{I}, x_j \in \Gamma_{i_j}^\times, i_j \neq i_{j+1} \right\}$$

the set of all infinite, alternating words, that is, in the context of Section 2.4 we have $V_\infty = \Gamma_\infty$.

By Proposition 2.10, the random walk converges again $\mathbb{P}_e$-a.s. to a random variable Z_∞ valued in Γ_∞. Denote by ν the distribution of Z_∞. Let

$$E_i := \left\{ x_1 x_2 \cdots \in \Gamma_\infty \;\middle|\; \tau(x_1) = i \right\} \text{ for } i \in \mathcal{I}.$$

Then ν is uniquely determined by the distribution of the Borel-sets B, where B is of the form $xE_i = \{xh \mid h \in E_i\}$ with $i \in \mathcal{I}$, $x \in \Gamma$ and $\tau(x) \neq i$. We will now give a formula for the distribution of these Borel-sets:

Lemma 3.21. *Let $i \in \mathcal{I}$, $x \in \Gamma$ with $\tau(x) \neq i$. Then*

$$\nu(xE_i) = \mathbb{P}_e[Z_\infty \in xE_i] = F(e,x) \cdot \Big(1 - \big(1 - \xi_i\big)\, G_i\big(e_i, e_i | \xi_i\big) \Big).$$

Proof. The proof of the lemma is extrapolated from Woess [40, Theorem 4c], where one can find an erroneous formula, which we correct here. First, we have

$$\nu(xE_i) = F(e,x|1) \cdot \nu(E_i).$$

Recall that we have by vertex-transitivity

$$G_i(o_i, o_i | z) = G_i(y, y | z)$$

for all $i \in \mathcal{I}$ and all $y \in \Gamma_i$. By Lemma 3.16, we obtain

$$\nu(E_i) = G(o,o|1) \cdot \rho(i) = 1 - (1 - \xi_i)\, G_i(e_i, e_i | \xi_i).$$

This leads to the proposed formula. $\square$

Now we reformulate our problem for finding a formula for ℓ. For this purpose, we apply a technique going back to Furstenberg [12], which was used by Ledrappier [21, Section 4b] for free groups.

By Lebesgue's Convergence Theorem, we have $\mathbb{P}_e - a.s.$

$$\lim_{n\to\infty} \frac{\mathbb{E}[\ell(Z_n)]}{n} = \lim_{n\to\infty} \int \frac{\ell(Z_n)}{n}\, d\mathbb{P}_e = \int \lim_{n\to\infty} \frac{\ell(Z_n)}{n} d\mathbb{P}_e = \int \ell \, d\mathbb{P}_e = \ell.$$

Thus, it is sufficient to prove convergence of the sequence

$$\Big(\mathbb{E}[\ell(Z_{n+1})] - \mathbb{E}[\ell(Z_n)] \Big)_{n \in \mathbb{N}}$$

and to compute its $\mathbb{P}_e$-a.s. limit, which then must equal ℓ. First, we have

$$\mathbb{E}[\ell(Z_n)] = \sum_{h\in\Gamma} \ell(h)\,\mu^{(n)}(h)$$

and

$$\mathbb{E}[\ell(Z_{n+1})] = \sum_{g,h\in\Gamma} \ell(gh)\,\mu(g)\,\mu^{(n)}(h).$$

On the other hand,

$$\mathbb{E}[\ell(Z_n)] = \sum_{g\in\Gamma} \mu(g)\,\mathbb{E}[\ell(Z_n)] = \sum_{g,h\in\Gamma} \mu(g)\,\ell(h)\,\mu^{(n)}(h).$$

Thus, we obtain

$$\begin{aligned}\mathbb{E}[\ell(Z_{n+1})] - \mathbb{E}[\ell(Z_n)] &= \sum_{g\in\Gamma}\mu(g)\sum_{h\in\Gamma}\big(\ell(gh)-\ell(h)\big)\,\mu^{(n)}(h)\\ &= \sum_{g\in\Gamma}\mu(g)\int\big(\ell(gZ_n)-\ell(Z_n)\big)\,d\mathbb{P}_e.\end{aligned}$$

Define now the random variables

$$Y_{g,n} := \ell(gZ_n) - \ell(Z_n)$$

for any given $g \in \bigcup_{i\in\mathcal{I}} \Gamma_i^\times$ and $n \in \mathbb{N}$. We have $Y_{g,n} \in \{-1,0,1\}$ for all $n \in \mathbb{N}$ as

$$\big|\ell(gZ_n) - \ell(Z_n)\big| \leq 1.$$

By vertex-transitivity, gZ_n converges to gZ_∞. Hence, $Y_{g,n}$ converges almost surely to a random variable $Y_{g,\infty}$ valued in $\{-1,0,1\}$ depending only on g and the first block of Z_∞. In other words, $Y_{g,n}$ rests constant for n big enough. Indeed, if $Z_\infty = x_1x_2\cdots \in \Gamma_\infty$, we obtain for given $g \in \bigcup_{i\in\mathcal{I}} \Gamma_i^\times$:

$$Y_{g,\infty} = \begin{cases} 0 & \text{, if } \tau(x_1) = \tau(g) \wedge x_1 g \neq e \\ -1 & \text{, if } \tau(x_1) = \tau(g) \wedge x_1 g = e \\ 1 & \text{, if } \tau(x_1) \neq \tau(g) \end{cases}.$$

By Lebesgue's Convergence Theorem, we infer that

$$\int\big(\ell(gZ_n)-\ell(Z_n)\big)\,d\mathbb{P}_e \xrightarrow{n\to\infty} \int Y_{g,\infty}\,d\mathbb{P}_e.$$

Consider the function

$$f : \Big(\bigcup_{i\in\mathcal{I}} \Gamma_i^\times\Big) \times \Gamma_\infty \to \{-1,0,1\}$$

defined by

$$(g, x_1 x_2 \dots) \mapsto \lim_{n\to\infty} \bigl(\ell(g x_1 \dots x_n) - \ell(x_1 \dots x_n)\bigr)$$

and its projections

$$f_g : \Gamma_\infty \to \{-1, 0, 1\} : w \mapsto f(g, w),$$

for every $g \in \bigcup_{i\in\mathcal{I}} \Gamma_i^\times$. Observe that each f_g is measureable and thus

$$\int Y_{g,\infty}\, d\mathbb{P}_e = \int_{\Gamma_\infty} f(g, Z_\infty)\, d\nu = \int_{\Gamma_\infty} f_g(w)\, d\nu(w).$$

Denote by E_h the event that Z_∞ has as first block the element $h \in \bigcup_{i\in\mathcal{I}} \Gamma_i^\times$ and denote by $E_{\neq i}$ the event that Z_∞ starts with a block element not of type $i \in \mathcal{I}$. Then we obtain for $g \in \Gamma_i^\times$

$$\nu(E_{g^{-1}}) = F(e, g^{-1}|1) \cdot \bigl(1 - \nu(E_i)\bigr) = F(e, g^{-1}|1) \cdot (1 - \xi_i) \cdot G_i(e_i, e_i|\xi_i)$$

and

$$\nu(E_{\neq i}) = 1 - \nu(E_i) = (1 - \xi_i) \cdot G_i(e_i, e_i|\xi_i).$$

For reason of better readability, we use again the short notation $G_i(\xi_i)$ for $G_i(e_i, e_i|\xi_i)$. Now we can conclude:

$$\begin{aligned}
& \mathbb{E}[\ell(Z_{n+1})] - \mathbb{E}[\ell(Z_n)] \\
= \quad & \sum_{g\in\Gamma} \mu(g) \int \bigl(\ell(gZ_n) - \ell(Z_n)\bigr)\, d\mathbb{P}_e \\
\xrightarrow{n\to\infty,\, a.s.} \quad & \sum_{g\in\Gamma^\times} \mu(g) \int_{\Gamma_\infty} f_g(w)\, d\nu(w) \\
= \quad & \sum_{i\in\mathcal{I}} \sum_{g\in\Gamma_i^\times} \mu(g) \bigl(-\nu(E_{g^{-1}}) + \nu(E_{\neq i})\bigr) \\
= \quad & \sum_{i\in\mathcal{I}} \sum_{g\in\Gamma_i^\times} \alpha_i\, \mu_i(g)\, (1 - \xi_i)\, G_i(\xi_i)\, \bigl(1 - F_i(e_i, g^{-1}|\xi_i)\bigr) \\
= \quad & \sum_{i\in\mathcal{I}} \alpha_i\, (1 - \xi_i)\, G_i(\xi_i)\, \bigl(1 - \underbrace{\sum_{g\in\Gamma_i^\times} \mu_i(g)\, F_i(e_i, g^{-1}|\xi_i)\bigr)}_{=\frac{G_i(\xi_i)-1}{\xi_i \cdot G_i(\xi_i)}} \\
= \quad & \sum_{i\in\mathcal{I}} \alpha_i\, \frac{1-\xi_i}{\xi_i} \bigl(1 - (1 - \xi_i)\, G_i(\xi_i)\bigr).
\end{aligned}$$

Thus, we get once more the rate of escape of the block length:

Corollary 3.22.

$$\ell = \sum_{i \in \mathcal{I}} \alpha_i \frac{1-\xi_i}{\xi_i} \bigl(1 - (1-\xi_i)\, G_i(\xi_i)\bigr).$$

□

We conclude this section with two final remarks:

1. Note that the probability measures do not need to be irreducible, since this property is not necessarily required in the proofs.
2. Observe that the presented technique in this section can be extended to a free product of an infinite, countable number of groups. All the necessary properties of the used generating functions from Section 2.3 hold also in this case. Furthermore, $Y_{g,n}$ is again bounded such that finiteness of ℓ is ensured. Thus, the same computations prove the same formula for ℓ if $\mathcal{I} = \mathbb{N}$.

3.4 Partial Rate of Escape w.r.t. the Block Length

We think of X again as a general free product of graphs, not necessarily equipped with a group structure as in the last section. We extend the considerations of Section 3.1 to the question at which frequency the random walk on the free product X visits different copies of a single graph X_i for some $i \in \mathcal{I}$. For this purpose, we define:

Definition 3.23 (Partial Block Length). *Let $x = x_1 \dots x_m \in V \setminus \{o\}$ and $i \in \mathcal{I}$. Then the i-th partial block length of x is*

$$\ell_i(x) := \operatorname{card}\bigl\{ j \in \{1, \dots, m\} \bigm| x_j \in X_i \bigr\}$$

and $\ell_i(o) := 0$.

We can now define the following characteristical 'speed' number for each $i \in \mathcal{I}$: if there is a constant $\ell_i \in \mathbb{R}_{\geq}$ such that

$$\ell_i := \lim_{n \to \infty} \frac{1}{n} \ell_i(Z_n) \text{ almost surely,}$$

then ℓ_i is called the i-th partial rate of escape w.r.t. the block length. Existence of ℓ_i for all $i \in \mathcal{I}$ can be easily shown using the considerations of Section 3.1.3. We have:

$$0 \leq \ell_i(Z_n) - \ell_i(W_{\mathbf{k}(n)}) \leq n - \mathbf{e}_{\mathbf{k}(n)} < \mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)},$$

as the maximal difference of the i-th partial block length between the words Z_n and $Z_{\mathbf{e}_{\mathbf{k}(n)}}$ equals $n - \mathbf{e}_{\mathbf{k}(n)}$. Using relation (3.5) from Section 3.1.3 we obtain

$$\frac{\ell_i(Z_n) - \ell_i(W_{\mathbf{k}(n)})}{n} \xrightarrow{n\to\infty} 0 \quad \mathbb{P}_o - a.s.. \tag{3.12}$$

Furthermore, we get:

Lemma 3.24. *Let $i \in \mathcal{I}$. Then*

$$\frac{1}{k}\ell_i(W_k) \xrightarrow{k\to\infty} \nu(i) \quad \mathbb{P}_o - a.s..$$

Proof. Consider the Markov chain $(\tau_k)_{k\in\mathbb{N}}$ on the finite state space $\mathcal{I}$ with its invariant probability measure ν. Define the indicator function on $\mathcal{I}$ w.r.t. $i \in \mathcal{I}$ as

$$\mathbb{1}_i : \mathcal{I} \to \{0,1\} : j \mapsto \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}.$$

By the Ergodic Theorem 3.6, we obtain

$$\frac{1}{k}\ell_i(W_k) = \frac{1}{k}\sum_{j=1}^{k} \mathbb{1}_i\big(\tau(W_j)\big) \xrightarrow{k\to\infty} \sum_{j\in\mathcal{I}} \mathbb{1}_i(j)\nu(j) = \nu(i) \quad \mathbb{P}_o - a.s..$$

□

Hence, we can conclude:

Corollary 3.25. *Let $i \in \mathcal{I}$. Then*

$$\frac{\ell_i(Z_n)}{n} \xrightarrow{n\to\infty} \nu(i)\cdot \ell = \frac{\nu(i)}{\Lambda} \quad \mathbb{P}_o - a.s..$$

Proof. Applying Corollary 3.13, Lemma 3.24, (3.6) and (3.12), we get $\mathbb{P}_o$-a.s.:

$$\begin{aligned} \frac{\ell_i(Z_n)}{n} &= \frac{\ell_i(Z_n) - \ell_i(W_{\mathbf{k}(n)})}{n} + \frac{\ell_i(W_{\mathbf{k}(n)})}{n} \\ &= \frac{\ell_i(Z_n) - \ell_i(W_{\mathbf{k}(n)})}{n} + \frac{\ell_i(W_{\mathbf{k}(n)})}{\mathbf{k}(n)} \cdot \frac{\mathbf{k}(n)}{\mathbf{e}_{\mathbf{k}(n)}} \cdot \frac{\mathbf{e}_{\mathbf{k}(n)}}{n} \xrightarrow{n\to\infty} \frac{\nu(i)}{\Lambda}. \end{aligned}$$

□

Remarks:

1. If $r = 2$, then we have $\nu(1) = \nu(2) = 1/2$. In this case we obviously have $\ell_1 = \ell_2 = \ell/2$.
2. Observe that $\ell = \sum_{i\in\mathcal{I}} \ell_i$.
3. The techniques of Sections 3.2 and 3.3 can also be adapted easily to compute not only ℓ, but also ℓ_i.

3.5 Deviation from the Limit Path

In this section, we want to link the random walk on the free product X with its boundary limit Z_∞, showing that the random walk is logarithmically quite close to the path induced by Z_∞. For this purpose, define for $x = x_1 \dots x_m \in V$ and $y = y_1 \dots y_n \in V$

$$x \wedge y := x_1 \dots x_k,$$

where $k = \max\bigl\{0 \leq j \leq \min\{m,n\} \mid x^{(j)} = y^{(j)}\bigr\}$, the confluent of x and y. If $x_1 \neq y_1$, then $x \wedge y = o$. Furthermore, define

$$\mathrm{dist}(x,y) := \ell(x) - \ell(x \wedge y).$$

For sets $S \subseteq V$ and $x \in V$ define

$$\mathrm{dist}(x,S) := \min\bigl\{\mathrm{dist}(x,s) \,|\, s \in S\bigr\}$$

and

$$\bar{Z}_\infty := \bigl\{o, Z_\infty^{(m)} \bigm| m \in \mathbb{N}\bigr\}.$$

We want to show:

Theorem 3.26. *Let $\theta := \max\{\xi_i \mid i \in \mathcal{I}\}$. Then we have almost surely*

$$\limsup_{n\to\infty} \frac{\mathrm{dist}(Z_n, \bar{Z}_\infty)}{\log n} \leq \frac{1}{-\log \theta}.$$

A similar theorem is given in a more special case of trees with finitely many cone types in Nagnibeda and Woess [30, Section 7], which is extrapolated from Ledrappier [21]. As the conclusion in [30] is proved in a wrong way, we give a complete proof for our case using the ideas of [30]. To prove Theorem 3.26, we need the following two lemmas:

Lemma 3.27. *For $i \in \mathcal{I}$ and $x, y \in V_i$ with $x \neq y$,*

$$F_i(x,y \,|\, \xi_i) \leq \xi_i.$$

Proof. Let $T_y^{(i)}$ be the first visit stopping time of $y \in V_i$ for the random walk P_i on X_i starting at $x \in V_i \setminus \{y\}$. As $\xi_i < 1$, we obtain:

$$\begin{aligned}
F_i(x,y \,|\, \xi_i) &= \sum_{n\geq 1} \mathbb{P}_x[T_y^{(i)} = n]\, \xi_i^n \\
&= \xi_i \cdot \sum_{n\geq 1} \mathbb{P}_x[T_y^{(i)} = n]\, \xi_i^{n-1} \\
&\leq \xi_i \cdot \sum_{n\geq 1} \mathbb{P}_x[T_y^{(i)} = n] \\
&= \xi_i \cdot F_i(x,y \,|\, 1) \\
&\leq \xi_i.
\end{aligned}$$

$\square$

Lemma 3.28. *For $r > 0$,*

$$\mathbb{P}_o\big[\mathrm{dist}(Z_n, \bar{Z}_\infty) \geq r\big] \leq \theta^r.$$

Proof. Let $x = x_1 \dots x_n \in V$. If $n < \lceil r \rceil$, then $\mathrm{dist}(x, \bar{Z}_\infty) < r$ and consequently $\mathbb{P}_x\big[\mathrm{dist}(x, \bar{Z}_\infty) \geq r\big] = 0$. If $n \geq r$, then $\mathrm{dist}(x, \bar{Z}_\infty) \geq r$ means that Z_∞ does not have $x_1 \dots x_{n-\lceil r \rceil+1}$ as a prefix. Equivalently, the length of the common prefix of x and Z_∞ is at most $n - \lceil r \rceil$. Observe that

$$F(x_1 \dots x_j, x_1 \dots x_{j-1}|1) = F_{\tau(x_j)}(x_j, o_{\tau(j)}|\xi_j) \quad \text{for all } j \in \{1, \dots, n\},$$

where $x_0 = o$. Writing $y := x_1 \dots x_{n-\lceil r \rceil}$ we obtain with Lemma 1.6 (iv), Proposition 2.5 and Lemma 3.27

$$\begin{aligned}
& \mathbb{P}_x\big[\mathrm{dist}(x, \bar{Z}_\infty) \geq r\big] \\
= \; & F(x, y) \cdot \mathbb{P}_y\big[Z_\infty^{(n-\lceil r \rceil+1)} \neq x_1 \dots x_{n-\lceil r \rceil+1}\big] \\
= \; & \prod_{j=n-\lceil r \rceil+1}^{n} F_{\tau(x_j)}\big(x_j, o_{\tau(x_j)} \mid \xi_{\tau(x_j)}\big) \cdot \mathbb{P}_y\big[Z_\infty^{(n-\lceil r \rceil+1)} \neq x_1 \dots x_{n-\lceil r \rceil+1}\big] \\
\leq \; & \theta^{\lceil r \rceil} \cdot \mathbb{P}_y\big[Z_\infty^{(n-\lceil r \rceil+1)} \neq x_1 \dots x_{n-\lceil r \rceil+1}\big] \leq \theta^r.
\end{aligned}$$

Now we can conclude:

$$\mathbb{P}_o\big[\mathrm{dist}(Z_n, \bar{Z}_\infty) \geq r\big] = \sum_{x \in V} \mathbb{P}_o[Z_n = x] \cdot \mathbb{P}_x\big[\mathrm{dist}(x, \bar{Z}_\infty) \geq r\big] \leq \theta^r.$$

□

Proof of Theorem 3.26. Let $\delta > 0$ and define the sequence of events

$$A_n := \left[\mathrm{dist}(Z_n, \bar{Z}_\infty) \geq \frac{\log n^{1+\delta}}{-\log \theta}\right].$$

Then $\mathbb{P}_o[A_n] \leq 1/n^{1+\delta}$ and, by the Borel-Cantelli lemma, with probability 1 only a finite number of these events occur. Thus, we have almost surely

$$\limsup_{n \to \infty} \frac{\mathrm{dist}(Z_n, \bar{Z}_\infty)}{(1+\delta)\log n} \leq \frac{1}{-\log \theta}.$$

Making δ arbitrarily small yields the claim. □

3.6 Sample Computations

This section gives sample computations for the rate of escape w.r.t. the block length of random walks on different free products of graphs. We show how to compute the required generating functions, and thus ℓ.

3.6.1 $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

Consider the groups

$$\Gamma_1 := \langle a \,|\, a^2 = e_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \Gamma_2 := \langle b \,|\, b^3 = e_2 \rangle \simeq \mathbb{Z}/3\mathbb{Z},$$

where e_1, e_2 respectively, denote the identity on Γ_1, on Γ_2 respectively. We are interested in a nearest neighbour random walk on the Cayley graph of $\Gamma_1 * \Gamma_2$ (see Figure 3.1), which is (using the notation from Section 3.3) governed by a probability measure μ with $\mu(b) = p$, $\mu(b^2) = q$ and $\mu(a) = 1-p-q$, where $0 \leq p \leq q < 1$ and $0 < p+q < 1$. This implies $\alpha_1 = 1-p-q$, $\alpha_2 = p+q$, $\mu_1(a) = 1$, $\mu_2(b) = p/(p+q)$ and $\mu_2(b^2) = q/(p+q)$.

The Cayley graph of the free product group of these two groups is not amenable. This ensures that the corresponding Green function $G(o, o|z)$ has radius of convergence bigger than 1. Our aim is now the computation of ℓ using Corollary 3.14.

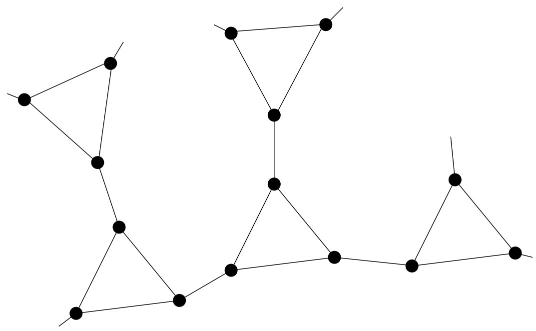


Figure 3.1: Structure of the Cayley graph of the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

We obtain the following generating functions:

$$\begin{aligned}
F_1(a, o_1|z) &= z, \qquad G_1(o_1, o_1|z) = \frac{1}{1-z^2}, \\
F_2(b, o_2|z) &= \sum_{m\geq 1} \mu_2(b)^{m+1}\mu_2(b^2)^{m-1} z^{2m} + \sum_{m\geq 0} \mu_2(b)^m \mu_2(b^2)^{m+1} z^{2m+1} \\
&= \sum_{m\geq 1} \Big(\frac{z}{p+q}\Big)^{2m} p^{m+1}q^{m-1} + \sum_{m\geq 0} \Big(\frac{z}{p+q}\Big)^{2m+1} p^m q^{m+1} \\
&= \frac{p}{q}\sum_{m\geq 0} \Big(\frac{\sqrt{pq}\,z}{p+q}\Big)^{2m} - \frac{p}{q} + \frac{qz}{p+q}\sum_{m\geq 0}\Big(\frac{\sqrt{pq}\,z}{p+q}\Big)^{2m} \\
&= \Big(\frac{p}{q} + \frac{qz}{p+q}\Big)\frac{1}{1-\Big(\frac{\sqrt{pq}\,z}{p+q}\Big)^2} - \frac{p}{q}.
\end{aligned}$$

Analogously, in the case $p \neq 0$ we get

$$F_2(b^2, o_2|z) = \left(\frac{q}{p} + \frac{pz}{p+q}\right)\frac{1}{1-\left(\frac{\sqrt{pq}\,z}{p+q}\right)^2} - \frac{q}{p}.$$

If $p = 0$ then $F_2(b^2, o_2|z) = z^2$. Consequently,

$$U_2(o_2, o_2|z) = \frac{pz}{p+q}F_2(b, o_2|z) + \frac{qz}{p+q}F_2(b^2, o_2|z)$$

and

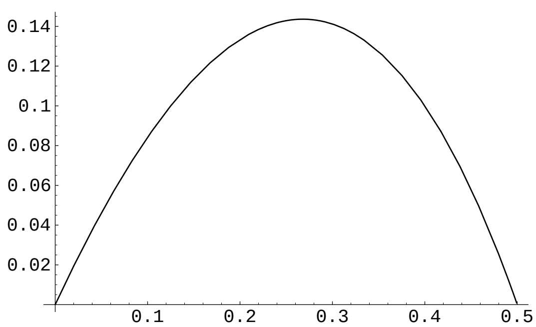
$$G_2(o_2, o_2|z) = \frac{1}{1 - U_2(o_2, o_2|z)}.$$

Furthermore,

$$\begin{aligned}
\bar{H}_1(z) &= p \cdot z \cdot F_2\big(b, o_2|\xi_2(z)\big) + q \cdot z \cdot F_2\big(b^2, o_2|\xi_2(z)\big),\\
\bar{H}_2(z) &= (1-p-q) \cdot z \cdot \xi_1(z),\\
\xi_1(z) &= \frac{(1-p-q) \cdot z}{1 - p \cdot z \cdot F_2\big(b, o_2|\xi_2(z)\big) - q \cdot z \cdot F_2\big(b^2, o_2|\xi_2(z)\big)},\\
\xi_2(z) &= \frac{(p+q) \cdot z}{1 - (1-p-q) \cdot z \cdot \xi_1(z)}.
\end{aligned}$$

By plugging $\xi_1(z)$ into $\xi_2(z)$, we obtain an equation in the variable $\xi_2(z)$. Solving this cubic equation with MATHEMATICA leads to three solutions, where two of these can be dropped, as $\xi_2(1) < 1$ has to hold and $\xi_2(z)$ has to be strictly increasing and continuous. Due to the complexity of this solution, we omit to state $\xi_2(z)$ explicitly.

As the block length $\ell(Z_n)$ goes to infinity, it must be $\hat{q}(1,2) = \hat{q}(2,1) = 1$, that is, $\nu(1) = \nu(2) = 1/2$. Now we are able to compute the constant Λ from Corollary 3.12 and by that ℓ. It is also possible to compute ℓ by the



p	q	ℓ
1/6	1/5	0.12803
1/4	1/3	0.14269
1/3	1/3	0.13333
1/4	1/2	0.11657
1/4	2/3	0.04989
1/8	3/4	0.07161

Figure 3.2: Rate of escape on $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

formulas given in the Corollaries 3.20 and 3.22. Figure 3.2 shows the values of ℓ for different values of p and q. The graph in Figure 3.2 illustrates the value of ℓ on the y-axis in dependence of $p = q$ on the x-axis.

Note that in this example the rate of escape w.r.t. the block length and the rate of escape w.r.t. the minimal path length are obviously identical if $p \neq 0$.

3.6.2 Free Product of Non-Cayley-Graphs

Let X_1, X_2 and X_3 be graphs as shown in Figure 3.3, with the sketched transition probabilities. Consider the corresponding random walk on the

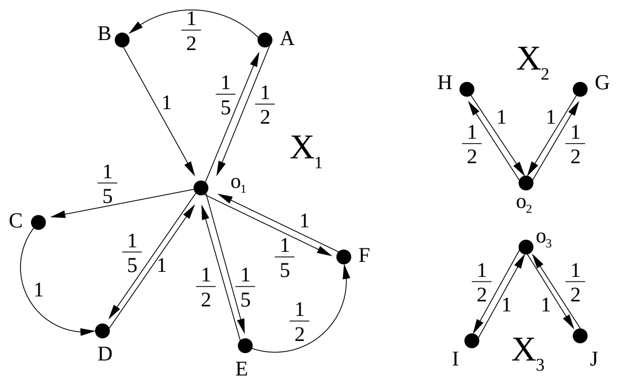


Figure 3.3: Some non-Cayley graphs

free product $X = X_1 * X_2 * X_3$, where $\alpha_1 = 5/9$ and $\alpha_2 = \alpha_3 = 2/9$. Note that none of X_1, X_2 and X_3 is a Cayley graph, and X also not. This example demonstrates the power of our formulas for ℓ, as we can even apply them on free products of non-transitive graphs. We now compute ℓ with the help of Corollary 3.14.

We obtain the following generating functions:

$$\begin{aligned}
U_1(o_1,o_1|z) &= \frac{1}{5}z\left(\frac{1}{2}z+\frac{1}{2}z^2+z^2+z+\frac{1}{2}z+\frac{1}{2}z^2+z\right) = \frac{3}{5}z^2+\frac{2}{5}z^3,\\
G_1(o_1,o_1|z) &= \frac{1}{1-U_1(o_1,o_1|z)} = \frac{1}{1-\frac{3}{5}z^2-\frac{2}{5}z^3},\\
F_1(A,o_1|z) &= F_1(E,o_1|z) = \frac{1}{2}z+\frac{1}{2}z^2,\\
F_1(C,o_1|z) &= z^2, \quad F_1(D,o_1|z) = F_1(F,o_1|z) = z,\\
G_2(o_2,o_2|z) &= \frac{1}{1-z^2}, \quad F_2(G,o_2,z) = F_2(H,o_2|z) = z,
\end{aligned}$$

$$\begin{aligned}
\bar{H}_1(z) &= 2 \cdot \frac{2}{9}\, z\, \xi_2(z) = \frac{4}{9}\, z\, \xi_2(z), \\
\bar{H}_2(z) &= \bar{H}_3(z) = \frac{2}{9}\, z\, \xi_2(z) + \sum_{s \in \mathcal{S}(o_1)} p(o,s) \cdot z \cdot F(s,o|z) \\
&= \frac{2}{9}\, z\, \xi_2(z) + \frac{5}{9} \cdot \frac{1}{5}\, z \left(3\xi_1(z) + 2\xi_1(z)^2\right).
\end{aligned}$$

Note that $\bar{H}_2(z) = \bar{H}_3(z)$ follows by symmetry. Consider

$$\begin{aligned}
\xi_1(z) &= \frac{\frac{5}{9} z}{1 - \bar{H}_1(z)} = \frac{\frac{5}{9} z}{1 - \frac{4}{9}\, z\, \xi_2(z)} \qquad \text{and} \\
\xi_2(z) &= \xi_3(z) = \frac{\frac{2}{9} z}{1 - \bar{H}_2(z)} = \frac{\frac{2}{9} z}{1 - \frac{2}{9}\, z\, \xi_2(z) - \frac{1}{9}\, z \left(3\xi_1(z) + 2\xi_1(z)^2\right)}.
\end{aligned}$$

By plugging $\xi_1(z)$ into $\xi_2(z)$, we have to solve the following equation in the variable $\xi_2(z)$:

$$\xi_2(z) = \frac{-2z(4z\xi_2(z) - 9)^2}{-729 + 810z\xi_2(z) - 288z^2\xi_2(z)^2 + 135z^2 - 60z^3\xi_2(z) + 32z^3\xi_2(z)^3 + 50z^3}$$

or equivalently,

$$32z^3\xi_2(z)^4 - 288z^2\xi_2(z)^3 + (810z - 28z^3)\xi_2(z)^2 + (-729 - 9z^2 + 50z^3)\xi_2(z) + 162z = 0.$$

Solving this equation with MATHEMATICA we obtain four continuous solutions, where only one solution fulfills $\xi_2(1) < 1$. Thus, we obtain $\xi_2(z)$ as this solution, and by that, we get $\xi_1(z)$. It is

$$\xi_1(1) \approx 0.66571 \quad \text{and} \quad \xi_2(1) = \xi_3(1) \approx 0.37231.$$

We get the transition matrix to the Markov chain of the alternating vertex types as

$$\big(\hat{q}(i,j)\big)_{1 \le i,j \le 3} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.62769 & 0 & 0.37231 \\ 0.62769 & 0.37231 & 0 \end{pmatrix},$$

and by that the invariant probability measure ν with

$$\nu(1) = 0.38563 \quad \text{and} \quad \nu(2) = \nu(3) = 0.30718.$$

Now we are able to compute the rate of escape w.r.t. the block length to the random walk on X. We obtain

$$\ell \approx 0.33089$$

and also

$$\ell_1 = \nu(1) \cdot \ell \approx 0.12760 \quad \text{and} \quad \ell_2 = \ell_3 = \nu(2) \cdot \ell \approx 0.10164.$$

If ℓ is computed by the formula given in Corollary 3.20, then the numerical approximated result and the above result coincide in the first 50 decimal numbers. Hence, numerical approximations do not necessarily lead to a distortion of the result in dependence which formula is used. We remark that the associated Green function $G(o, o|z)$ has radius of convergence bigger than 1: this can be verified with the help of the computed generating functions.

3.6.3 $\mathbb{Z}^2 * \mathbb{Z}^2$ and $\mathbb{Z}^2 * \mathbb{Z}/2\mathbb{Z}$

Let X_1 be the Cayley graph of $\mathbb{Z}^2 = \langle (\pm 1, 0), (0, \pm 1) \rangle$, X_1' another copy of it and let X_2 be the Cayley graph of the group $\mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 = e_2 \rangle$, where e_2 is the identity on $\mathbb{Z}/2\mathbb{Z}$. The simple random walk on $\mathbb{Z}$ is governed by the probability measure μ_1 with

$$\mu_1\big((\pm 1, 0)\big) = \mu_1\big((0, \pm 1)\big) = 1/4.$$

Using the notation from Section 3.3 we now consider the simple random walks on

I: $X = X_1 * X_1'$, that is, we consider the simple random walks on both X_1 and X_1', each governed by μ_1, and we set $\alpha_1 = \alpha_2 = \frac{1}{2}$.

II: $X = X_1 * X_2$, that is, μ_1 governs the simple random walk on X_1 and we set $\alpha_1 = \frac{4}{5}$, $\alpha_2 = \frac{1}{5}$ and $\mu_2(a) = 1$.

For the computation of the rate of escape w.r.t the block length of both random walks we use the formula given in Corollary 3.22. Therefore it is sufficient to compute ξ_1, ξ_2 and $G_1\big((0,0), (0,0)|\xi_1\big)$. For this purpose, we use the computations and results from Woess [43, pages 100, 104, 105 and 109].

Before we can compute the requested values, we have to introduce some new functions. Therefore let G be an oriented graph with root vertex o_{G}, on which we discuss an irreducible random walk. We write

$$W_{\mathsf{G}}(z) := z \cdot G_{\mathsf{G}}(o, o|z).$$

As $W_{\mathsf{G}}(z)$ is strictly increasing for $z \geq 0$, there is an inverse function $V_{\mathsf{G}}(z)$ with $V_{\mathsf{G}}\big(W_{\mathsf{G}}(z)\big) = z$. By [43, Theorem 9.10], we have

$$G_{\mathsf{G}}(o, o|z) = \Phi_{\mathsf{G}}\big(z\, G_{\mathsf{G}}(o, o|z)\big), \quad \text{where } \Phi_{\mathsf{G}}(t) := \frac{t}{V_{\mathsf{G}}(t)}.$$

Write now $W_1 = W_{X_1}$, $V_1 = V_{X_1}$, $\Phi_i = \Phi_{X_i}$ for $i \in \{1,2\}$ and $W = W_X$, $V = V_X$ and $\Phi = \Phi_X$. By [43, Example 9.15 (3)],

$$W_1(z) = \frac{1}{(2\pi)^2} \int_{(-\pi,\pi]^2} \frac{2z}{2 - z \cdot (\cos x_1 + \cos x_2)} d\underline{x},$$

where $\underline{x} = (x_1, x_2)$.

I. Consider now $X = \mathbb{Z}^2 * \mathbb{Z}^2$. By [43, Theorem 9.19] and symmetry, we have

$$\Phi(t) = 2\Phi_1(t/2) - 1.$$

This provides for real $z \geq 0$

$$G(o,o|z) = 2 \cdot \Phi_1\left(\frac{1}{2}W(z)\right) - 1,$$

or equivalently,

$$\Phi_1\left(\frac{1}{2}W(z)\right) = \frac{1}{2}\big(G(o,o|z) + 1\big).$$

Inverting this equation and multiplying with $\frac{1}{2}W(z)$ yields

$$\frac{\frac{1}{2}W(z)}{\Phi_1\left(\frac{1}{2}W(z)\right)} = \frac{W(z)}{G(o,o|z) + 1}.$$

Applying W_1 onto both sides of this equation yields

$$\frac{1}{2}W(z) = W_1\left(\frac{W(z)}{G(o,o|z) + 1}\right).$$

Write $G(z) := G(o,o|z)$ and $G_1(z) := G_1\big((0,0),(0,0)|z\big)$. By [43, Equation 9.20], we have

$$\xi_1(z)\, G_1\big(\xi_1(z)\big) = \alpha_1\, z\, G(z), \tag{3.13}$$

or equivalently,

$$W_1\big(\xi_1(z)\big) = \frac{1}{2}W(z). \tag{3.14}$$

If $u(z)$ is another function with $W_1\big(u(z)\big) = \frac{1}{2}W(z)$, then

$$u(z) = V_1\Big(W_1\big(u(z)\big)\Big) = V_1\left(\frac{1}{2}W(z)\right) = V_1\Big(W_1\big(\xi_1(z)\big)\Big) = \xi_1(z).$$

Hence,

$$\xi_1 = \xi_1(1) = \frac{W(1)}{G(o,o|1) + 1} = \frac{W(1)}{W(1) + 1}.$$

Substituting $y = \frac{1}{2}W(1)$, we get

$$\xi_1 = \frac{2y}{2y + 1},$$

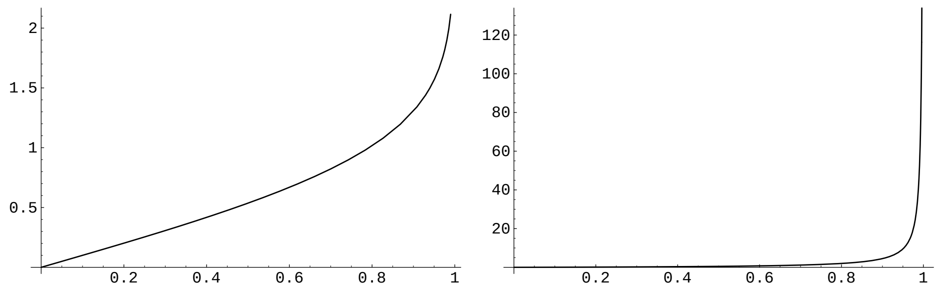
or equivalently,

$$y = \frac{\xi_1}{2(1-\xi_1)}.$$

By plugging y into equation (3.14), we have to solve

$$\frac{\xi_1}{2(1-\xi_1)} = W_1(\xi_1) = \frac{1}{4\pi^2}\int_{(-\pi,\pi]^2} \frac{2\xi_1}{2-\xi_1(\cos x_1 + \cos x_2)} d\underline{x}$$

in the still unknown variable ξ_1. The solution can not be computed explicitely, but numerically. Considering the graphs of the functions $W_1(z)$ and $z/(2(1-z))$ we see that there is only one intersection point, which must equal ξ_1 (see Figure 3.4). By using the bisection method with numerical integration and evaluation, we can isolate the solution at any accuracy. We obtain



(a) $W_1(z)$ (b) $\frac{z}{2(1-z)}$

Figure 3.4: Graphs of $W_1(z)$ and $\frac{z}{2(1-z)}$

$$\xi_1 \approx 0.54052.$$

Thus $y \approx 0.58819$ and

$$G(o,o|1) = W(1) = 2y \approx 1.17637.$$

With equation (3.13) we obtain

$$G_1\bigl(\xi_1(1)\bigr) = \frac{\frac{1}{2}G(o,o|1)}{\xi_1(1)} \approx 1.08819.$$

By symmetry, we have obviously $\xi_1 = \xi_2$ and $G_1(z) = G_2(z)$. Now we can apply the formula given in Corollary 3.22 for the computation of the rate of escape ℓ w.r.t. the block length of the random walk on $\mathbb{Z}^2 * \mathbb{Z}^2$. We obtain

$$\ell \approx 0.42503.$$

II. Consider now the free product $X = \mathbb{Z}^2 * \mathbb{Z}/2\mathbb{Z}$ with the corresponding simple random walk. We proceed as in the case of $\mathbb{Z}^2 * \mathbb{Z}^2$. By [43, Theorem 9.19], we have the equation

$$\Phi(t) = \Phi_1(\alpha_1 t) + \Phi_2(\alpha_2 t) - 1$$

and by [43, Example 9.15 (1)], it is

$$\Phi_2(t) = \frac{1}{2}\Big(\sqrt{1+4t^2}+1\Big).$$

This yields

$$\begin{aligned} G(o,o|z) &= \Phi_1\Big(\frac{4}{5}W(z)\Big) + \Phi_2\Big(\frac{1}{5}W(z)\Big) - 1 \\ &= \Phi_1\Big(\frac{4}{5}W(z)\Big) + \frac{1}{2}\bigg(\sqrt{1+\frac{4}{25}W(z)^2}-1\bigg), \end{aligned}$$

or equivalently,

$$\Phi_1\Big(\frac{4}{5}W(z)\Big) = G(o,o|z) - \frac{1}{2}\bigg(\sqrt{1+\frac{4}{25}W(z)^2}-1\bigg).$$

Inverting this equation and multiplication with $4\,W(z)/5$ leads to

$$\frac{\frac{4}{5}W(z)}{\Phi_1\big(\frac{4}{5}W(z)\big)} = \frac{\frac{4}{5}W(z)}{G(o,o|z) - \frac{1}{2}\Big(\sqrt{1+\frac{4}{25}W(z)^2}-1\Big)}.$$

By definition of Φ_1, an application of W_1 onto both sides of this equation yields

$$\frac{4}{5}W(z) = W_1\left(\frac{\frac{4}{5}W(z)}{G(o,o|z) - \frac{1}{2}\Big(\sqrt{1+\frac{4}{25}W(z)^2}-1\Big)}\right).$$

Note that we have by [43, Equation 9.20]

$$W_1\big(\xi_1(z)\big) = \frac{4}{5}W(z). \tag{3.15}$$

Analogously to the case $\mathbb{Z}^2 * \mathbb{Z}^2$ the following equation has to hold:

$$\xi_1(1) = \frac{\frac{4}{5}W(1)}{W(1) - \frac{1}{2}\Big(\sqrt{1+\frac{4}{25}W(1)^2}-1\Big)}.$$

Substituting $y = \frac{4}{5}W(1)$, we obtain

$$\xi_1 = \xi_1(1) = \frac{y}{\frac{5y}{4} - \frac{1}{2}\Big(\sqrt{1+\frac{1}{4}y^2}-1\Big)},$$

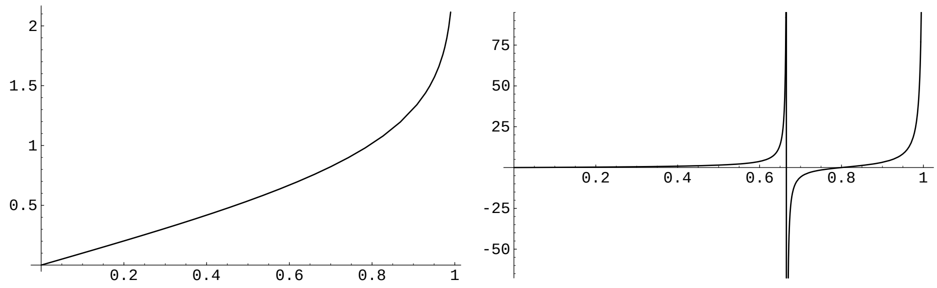
or equivalently,

$$y = \frac{(4 - 5\,\xi_1)\;\xi_1}{4 - 10\,\xi_1 + 6\,\xi_1^2}.$$

Plugging y into equation (3.15), we have to solve

$$\frac{(4 - 5\,\xi_1)\;\xi_1}{4 - 10\,\xi_1 + 6\,\xi_1^2} = W_1(\xi_1)$$

in the unknown variable ξ_1. Observe that $\xi_1 \geq 4/5$. Again, the solution can be computed only numerically. Considering the graphs of the functions $(4-5z)z/(4-10z+6z^2)$ and $W_1(z)$, we see that there is only one possible intersection point bigger than $\frac{4}{5}$. See Figure 3.5.



(a) $W_1(z)$ (b) $\frac{(4-5z)z}{4-10z+6z^2}$

Figure 3.5: Graphs of $W_1(z)$ and $\frac{(4-5z)z}{4-10z+6z^2}$

Using the bisection method and numerical integration and evaluation we obtain

$$\xi_1 \approx 0.84426$$

and

$$y \approx 1.12585$$

and

$$W(1) = \frac{5}{4} y \approx 1.40731.$$

This yields

$$G_1\big((0,0),(0,0)|\xi_1\big) = \frac{\frac{4}{5}W(1)}{\xi_1} \approx 1.33353.$$

Note that

$$\xi_1 = \frac{\frac{4}{5}}{1 - \frac{1}{5}\xi_2(1)}.$$

Thus,

$$\xi_2 = \xi_2(1) \approx 0.26212$$

and

$$G_2(e_2, e_2|\xi_2) = \frac{1}{1-\xi_2^2} \approx 1.07378.$$

Now we have computed all required values. The rate of escape w.r.t. the block length of the simple random walk on $\mathbb{Z}^2 * \mathbb{Z}/2\mathbb{Z}$ is

$$\ell \approx 0.23385.$$

3.7 Summary

In this chapter we have presented three different techniques for the computation of the rate of escape ℓ: a probabilistic approach with a detailed investigation of the behaviour of the random walk, then an approach using double generating functions and an application of a theorem of Sawyer and Steger, and finally an approach using limit processes for random walks on free products of finite or countable groups. All three different techniques go beyond previous investigated classes of free products. In our case, we are not only restricted to (infinite) Cayley graphs, but we can also handle with free products of any graphs, if the Green functions to the corresponding random walks have radii of convergence bigger than 1. We extended our techniques for the computation of the i-th partial rate of escape w.r.t. the block length. Sample computations have shown how to compute the rate of escape explicitly.

The three obtained formulas for ℓ have different requirements. As the formulas given in the Corollaries 3.14 and 3.20 deal with derivatives, one has to know the explicit form of the Green functions on the factors X_i for all $i \in \mathcal{I}$, and also the associated functions $\xi_i(z)$. The formula given in Corollary 3.22, however, requires only the knowledge of the values $\xi_i(1)$ for all $i \in \mathcal{I}$ and also the evaluation results $G_i(o_i, o_i|\xi_i)$ of the Green functions. But recall that the latter formula can only be applied to the case of random walks on free products of groups. In general, however, it is very difficult to compute the required generating functions and values.

Chapter 4

Rate of Escape w.r.t. other Length Functions

Considering a random walk on the free product X in the sense of Chapter 2 we investigate in this chapter the rate of escape w.r.t. to other length functions beyond the block length. For this purpose, we extend the considerations of Sections 3.1 and 3.2. The technique of Section 3.3, though, is not suitable for such an extension. The plan for this chapter is as follows: in Section 4.1, we show existence of the rate of escape w.r.t. the minimal path length and compute a formula for this constant. In Section 4.2, we show existence of the almost sure, constant limit $\lim_{n\to\infty} l(Z_n)/n$ w.r.t. a length function l, which arises from bounded length functions on the factors X_i. Analogously to the previous chapter, we compute the partial rate of escape w.r.t. the minimal path length in Section 4.3, where we only respect lengths coming from the copies of a single factor X_i with some fixed $i \in \mathcal{I}$. Finally, we conclude this chapter by presenting sample computations in Section 4.4.

4.1 Computation by First Exit Times

In this section we use the considerations of Section 3.1 to prove almost sure convergence of the sequence of random variables $|Z_n|/n$ to a constant λ and to compute a formula for it. Recall that $|x|$, $x \in V$, is the minimal length of a path from the root o to x. In general, this is not the length of a shortest path from x to o (compare with Section 3.6.2). We introduce some notation. Let $i \in \mathcal{I}$ and $n \in \mathbb{N}$. Then the V_i-**ball** of radius n centered at o_i is given by $B_i(n) := \bigl\{x \in V_i \bigm| |x| \leq n\bigr\}$. The **sphere** of V_i with radius n is the set $S_i(n) = \bigl\{x \in V_i \bigm| |x| = n\bigr\}$.

As seen in Section 3.1.2, $\bigl(\widetilde{W}_k, \mathbf{i}_k, \tau_k\bigr)_{k\in\mathbb{N}}$ is a positive recurrent Markov chain on $\mathcal{A}$ with the invariant probability measure π given by equation (3.3).

Consider

$$h : \mathcal{A} \to \mathbb{N} : (y, n, j) \mapsto |y| .$$

We show that h is π-integrable:

Lemma 4.1. *$\sigma := \int h\, d\pi$ exists.*

Proof. Recall from Section 3.1.3 the definition of $g(y, n, j) = n$, where $(y, n, j) \in \mathcal{A}$. Observe that $(y, n, j) \in \mathcal{A}$ implies $n \geq |y|$, as the random walk on X is of nearest neighbour type. By Lemma 3.10,

$$\int h\, d\pi = \sum_{(y,n,j)\in\mathcal{A}} |y| \cdot \pi(y, n, j) \leq \sum_{(y,n,j)\in\mathcal{A}} n \cdot \pi(y, n, j) = \int g\, d\pi < \infty.$$

□

Observe that

$$\frac{1}{n}\sum_{k=1}^{n} h\big(\widetilde{W}_k, \mathbf{i}_k, \tau_k\big) = \frac{\big|\widetilde{W}_1\big| + \big|\widetilde{W}_2\big| + \cdots + \big|\widetilde{W}_n\big|}{n} = \frac{\big|W_n\big|}{n}.$$

An application of the *Ergodic Theorem for positive recurrent Markov chains*, Theorem 3.6, yields

$$\frac{1}{n}\sum_{k=1}^{n} h\Big(\widetilde{W}_k, \mathbf{i}_k, \tau_k\Big) \quad \xrightarrow{n\to\infty} \quad \int h\, d\pi = \sigma \quad \mathbb{P}_o - a.s.,$$

that is, $|W_n|/n$ converges to σ almost surely. Now

$$0 \leq |Z_n| - |W_{\mathbf{k}(n)}| \leq n - \mathbf{e}_{\mathbf{k}(n)} < \mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)} ,$$

as $|Z_n|$ can increase from time $\mathbf{e}_{\mathbf{k}(n)}$ on maximally by $n - \mathbf{e}_{\mathbf{k}(n)}$. This implies that $(|Z_n| - |W_{\mathbf{k}(n)}|)/n$ converges to zero; compare with Section 3.1.3. Using Corollary 3.13 and (3.6) we can prove analogously to the proof of Corollary 3.14:

Corollary 4.2.

$$\frac{|Z_n|}{n} \quad \xrightarrow{n\to\infty} \quad \lambda = \frac{\sigma}{\Lambda} = \sigma \cdot \ell \quad \mathbb{P}_o - a.s.$$

□

Finally, we want to compute a formula for σ. Therefore we define for $i \in \mathcal{I}$ and $M \subseteq V_i$ a modified Green function:

$$G_i(o_i, M|z) := \sum_{x\in M} G_i(o_i, x|z) = \sum_{n\geq 0} p_i^{(n)}(o_i, M)\, z^n,$$

where $p_i^{(n)}(o_i, M)$ is the probability for the random walk P_i on X_i starting at o_i to stand at any $y \in M$ after n steps. The integral $\int h\, d\pi$ can be rewritten as follows:

Lemma 4.3.

$$\sigma = \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \nu(i)\frac{\alpha_j}{\alpha_i}\frac{\xi_i}{\xi_j}\frac{1-\xi_j}{1-\xi_i}\sum_{m\geq 1}\sum_{y\in S_j(m)} m\cdot L_j(o_j,y|\xi_j)$$

Proof. Writing $G_i(\xi_i) := G_i(o_i,o_i|\xi_i)$, we obtain

$$\begin{aligned}
\sigma &= \sum_{(y,n,j)\in\mathcal{A}} |y|\cdot\sum_{i\in\mathcal{I}}\nu(i)\cdot q\bigl((x,m,i),(y,n,j)\bigr)\\
&= \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \nu(i)\alpha_j\frac{1-\xi_j}{1-\xi_i}\sum_{y\in V_j^\times}\sum_{s\in\mathcal{P}(y)} p_j(s,y)\cdot|y|\cdot\sum_{n\geq 1}k_i^{(n-1)}(o,s)\\
&= \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \nu(i)\alpha_j\frac{1-\xi_j}{1-\xi_i}\sum_{m\geq 1}\sum_{y\in S_j(m)}\sum_{s\in\mathcal{P}(y)} p_j(s,y)\cdot m\cdot\frac{L_j(o_j,s|\xi_j)}{1-\bar{H}_i(1)}\\
&= \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \nu(i)\frac{1-\xi_j}{1-\xi_i}\cdot\frac{\alpha_j}{G_j(\xi_j)}\cdot\sum_{m\geq 1}\sum_{y\in S_j(m)}\sum_{s\in\mathcal{P}(y)} m\cdot\frac{p_j(s,y)\cdot G_j(o_j,s|\xi_j)}{1-\bar{H}_i(1)}\\
&= \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \nu(i)\frac{\alpha_j}{\alpha_i}\frac{1-\xi_j}{1-\xi_i}\frac{1}{G_j(\xi_j)}\frac{\alpha_i}{\bigl(1-\bar{H}_i(1)\bigr)\cdot\xi_j}\sum_{m\geq 1}\sum_{y\in S_j(m)} m\cdot G_j(o_j,y|\xi_j)\\
&= \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \nu(i)\frac{\alpha_j}{\alpha_i}\frac{\xi_i}{\xi_j}\frac{1-\xi_j}{1-\xi_i}\sum_{m\geq 1}\sum_{y\in S_j(m)} m\cdot L_j(o_j,y|\xi_j).
\end{aligned}$$

□

We can also write:

$$\begin{aligned}
\sigma &= \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \nu(i)\frac{\alpha_j}{\alpha_i}\frac{\xi_i}{\xi_j}\frac{1-\xi_j}{1-\xi_i}\frac{1}{G_j(\xi_j)}\sum_{m\geq 1} m\cdot\sum_{k\geq 0}p_j^{(k)}\bigl(o_j,S_j(m)\bigr)\xi_j^k\\
&= \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \nu(i)\frac{\alpha_j}{\alpha_i}\frac{\xi_i}{\xi_j}\frac{1-\xi_j}{1-\xi_i}\frac{1}{G_j(\xi_j)}\sum_{m\geq 0}\Bigl(\frac{1}{1-\xi_j}-G_j\bigl(o_j,B_j(m)|\xi_j\bigr)\Bigr).
\end{aligned}$$

4.2 Computation by Double Generating Functions

Let $l(\cdot)$ be a length function on the free product X arising from bounded length functions $l_i(\cdot)$ on the factors X_i. Extending the considerations of Section 3.2, we show existence of a constant $\mathtt{l}\in\mathbb{R}$ such that

$$\mathtt{l} = \lim_{n\to\infty}\frac{1}{n}l(Z_n) \quad \text{almost surely,}$$

and we will compute a formula for $\mathtt{l}$. Obviously, the length functions $l_i(\cdot)$ are bounded, if all factors X_i are finite, that is, $\operatorname{card} V_i < \infty$ for each $i \in \mathcal{I}$. By Theorem 3.17, it is sufficient to consider the double generating function $\mathcal{E}(w,z)$ given by equation (3.9). Define now $S_i(n) := \{y \in V_i \mid l(y) = n\}$ for $i \in \mathcal{I}$ and $n \in \mathbb{N}$. Then we obtain:

Lemma 4.4. *Let $w, z \in \mathbb{R}$ with $0 < w, z < 1$ and let*

$$g(w,z) := 1 - \sum_{i \in \mathcal{I}} \frac{\sum_{n \geq 1} G_i\bigl(o_i, S_i(n) | \xi_i(z)\bigr)\, w^n}{G_i\bigl(o_i, o_i | \xi_i(z)\bigr) + \sum_{n \geq 1} G_i\bigl(o_i, S_i(n) | \xi_i(z)\bigr)\, w^n}.$$

Then

$$\mathcal{E}(w,z) = \frac{G(o,o|z)}{g(w,z)}.$$

Proof. Let $w, z \in \mathbb{R}$ with $0 < w, z < 1$. Recall that by Corollary 3.19 we have

$$\mathcal{E}(w,z) = \frac{G(o,o|z)}{1 - \sum_{i \in \mathcal{I}} \frac{\mathcal{L}_i^+(w,z)}{1 + \mathcal{L}_i^+(w,z)}},$$

where $\mathcal{L}_i^+(w,z) = \sum_{x \in V_i^\times} L_i\bigl(o_i, x | \xi_i(z)\bigr)\, w^{l(x)}$. This yields

$$\begin{aligned}
\frac{\mathcal{L}_i^+(w,z)}{1 + \mathcal{L}_i^+(w,z)} &= \frac{\sum_{x \in V_i^\times} L_i\bigl(o_i, x | \xi_i(z)\bigr)\, w^{l(x)}}{1 + \sum_{x \in V_i^\times} L_i\bigl(o_i, x | \xi_i(z)\bigr)\, w^{l(x)}} \\
&= \frac{\sum_{x \in V_i^\times} G_i\bigl(o_i, x | \xi_i(z)\bigr)\, w^{l(x)}}{G_i\bigl(o_i, o_i | \xi_i(z)\bigr) + \sum_{x \in V_i^\times} G_i\bigl(o_i, x | \xi_i(z)\bigr)\, w^{l(x)}} \\
&= \frac{\sum_{n \geq 1} G_i\bigl(o_i, S_i(n) | \xi_i(z)\bigr)\, w^n}{G_i\bigl(o_i, o_i | \xi_i(z)\bigr) + \sum_{n \geq 1} G_i\bigl(o_i, S_i(n) | \xi_i(z)\bigr)\, w^n},
\end{aligned}$$

and thus the proposed equation follows. $\square$

Note that we have assumed that all $l_i(\cdot)$ are bounded such that $S_i(n) \neq \varnothing$ only for finitely many $n \in \mathbb{N}$. Hence, there is some $\delta > 0$ such that $g(w,z)$ and $C(w,z) := G(o,o|z)$ are analytic, if $|w-1|, |z-1| < \delta$. This is due to the fact that $G(o,o|z)$ and $\xi_i(z)$ are continuous and have radii of convergence bigger than 1 and $\xi_i(1) < 1$. Furthermore $C(1,1) = G(o,o|1) \neq 0$, so all required conditions for an application of Theorem 3.17 are fulfilled. The partial derivatives of g w.r.t. to w and z evaluated at $(1,1)$ are again denoted by g_w and g_z. An application of Theorem 3.17 yields:

Corollary 4.5.

$$\frac{l(Z_n)}{n} \xrightarrow{n \to \infty} \mathtt{l} = \frac{g_w(1,1)}{g_z(1,1)} \quad \mathbb{P}_o - a.s.$$

$\square$

For reason of better readability, we write for $i \in \mathcal{I}$

$$\mathcal{G}_i(z) := \sum_{n\geq 1} n \cdot G_i\big(o_i, S_i(n)\big|\xi_i(z)\big).$$

Simplifications yield the following formula for the derivatives:

$$\begin{aligned} g_w(1,1) &= -\sum_{i\in\mathcal{I}} (1-\xi_i)^2 \cdot G_i(\xi_i) \cdot \mathcal{G}_i(1) \quad \text{and} \\ g_z(1,1) &= -\sum_{i\in\mathcal{I}} \xi_i'(1) \cdot \Big(G_i(\xi_i) - (1-\xi_i)\cdot G_i'(\xi_i)\Big), \end{aligned}$$

where $G_i(\xi_i) = G_i(o_i, o_i|\xi_i)$ and $G_i'(\xi_i)$ is the derivative of $G_i(o_i, o_i|z)$ w.r.t. z evaluated at ξ_i.

Remark: If one of the $l_i(\cdot)$ is not bounded, then convergence of $g(w,z)$ is possible for $w > 1$, but not ensured.

4.3 Partial Rate of Escape w.r.t. Minimal Path Length

We extend the considerations of Section 3.4 to the question which part of the rate of escape w.r.t. the minimal path length is provided by visits in copies of a single graph X_i for some $i \in \mathcal{I}$. Therefore we define:

Definition 4.6 (Partial Minimal Path Length). *Let $x = x_1 \dots x_n \in V$, $x \neq o$, $i \in \mathcal{I}$, and*

$$J(x) := \big\{x_j \mid j \in \{1,\dots,n\}, x_j \in X_i\big\}.$$

Then the partial minimal path length of x w.r.t. X_i is

$$|x|_i := \sum_{y\in J(x)} |y| \quad \text{and} \quad |o|_i := 0.$$

If there is a constant $\lambda_i \in \mathbb{R}_{\geq}$ such that

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} |Z_n|_i \ \text{ almost surely,}$$

then λ_i is called the i-th partial rate of escape w.r.t. the minimal path length. For our further computations we will need the following lemma:

Lemma 4.7. *Let*

$$h_i : \mathcal{A} \to \mathbb{N}_0 : (y,n,j) \mapsto |y|_i.$$

Then h_i is π-integrable, that is, $\sigma_i := \int h_i \, d\pi < \infty$.

Proof. Recall the definition of the function h from Section 4.1. Lemma 4.1 proves the claim, as $h(y,n,j) \geq h_i(y,n,j)$. $\square$

Existence of λ_i for all $i \in \mathcal{I}$ can be easily shown using the considerations of Section 3.1.3:

$$0 \leq |Z_n|_i - |W_{\mathbf{k}(n)}|_i \leq n - \mathbf{e}_{\mathbf{k}(n)} < \mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)},$$

because the maximal difference of the partial minimal path length w.r.t. X_i between the words Z_n and $Z_{\mathbf{e}_{\mathbf{k}(n)}}$ equals $n - \mathbf{e}_{\mathbf{k}(n)}$. Using (3.5) from Section 3.1.3 we obtain

$$\frac{|Z_n|_i - |W_{\mathbf{k}(n)}|_i}{n} \xrightarrow{n\to\infty} 0 \quad \mathbb{P}_o - a.s.. \tag{4.1}$$

Furthermore, we get:

Lemma 4.8. *Let $i \in \mathcal{I}$. Then*

$$\frac{1}{k}|W_k|_i \xrightarrow{k\to\infty} \sigma_i \quad \mathbb{P}_o - a.s..$$

Proof. Applying the ergodic theorem for positive recurrent Markov chains, Theorem 3.6, we obtain:

$$\frac{1}{k}|W_k|_i = \frac{1}{k}\sum_{j=1}^{k} \big|\widetilde{W}_j\big|_i \xrightarrow{k\to\infty} \int h_i \, d\pi = \sigma_i \quad \mathbb{P}_o - a.s..$$

$\square$

Analogously to the proof of Corollary 3.14, we get:

Corollary 4.9. *For $i \in \mathcal{I}$,*

$$\frac{|Z_n|_i}{n} \xrightarrow{n\to\infty} \lambda_i = \frac{\sigma_i}{\Lambda} = \sigma_i \cdot \ell \quad \mathbb{P}_o - a.s..$$

$\square$

Finally, we want to state a formula for σ_i:

Lemma 4.10. *For $i \in \mathcal{I}$,*

$$\sigma_i = \sum_{j \in \mathcal{I}\setminus\{i\}} \nu(j) \frac{\alpha_i}{\alpha_j} \frac{\xi_j}{\xi_i} \frac{1-\xi_i}{1-\xi_j} \sum_{m\geq 1} \sum_{y \in S_i(m)} m \cdot L_i(o_i, y|\xi_i).$$

Proof. We obtain analogously to the proof of Lemma 4.3:

$$\begin{aligned}\sigma_i &= \sum_{(y,n,i)\in\mathcal{A}} |y|_i \cdot \sum_{j\in\mathcal{I}} \nu(j)\, q\bigl((x,m,j),(y,n,i)\bigr)\\ &= \sum_{j\in\mathcal{I}\setminus\{i\}} \nu(j) \frac{\alpha_i}{\alpha_j} \frac{\xi_j}{\xi_i} \frac{1-\xi_i}{1-\xi_j} \sum_{m\geq 1} \sum_{y\in S_i(m)} m \cdot L_i(o_i, y|\xi_i).\end{aligned}$$

□

Remarks:

1. Obviously we have $\sum_{i\in\mathcal{I}} \sigma_i = \sigma$ and $\sum_{i\in\mathcal{I}} \lambda_i = \lambda$.
2. If X_i is finite, Section 4.2 provides another formula for the limit λ_i: setting $l_i(x) := |x|$ for $x \in V_i$ and $l_j(y) = 0$ for all $j \in \mathcal{I}\setminus\{i\}$ and $y \in V_j$ yields

$$\lambda_i = \frac{(1-\xi_i)^2 \cdot G_i(\xi_i) \cdot \mathcal{G}_i(1)}{\sum_{i\in\mathcal{I}} \xi_i'(1) \cdot \Bigl(G_i(\xi_i) - (1-\xi_i)\cdot G_i'(\xi_i)\Bigr)}.$$

4.4 Sample Computations

4.4.1 Free Product of Non-Cayley-Graphs

The rate of escape w.r.t. the minimal path length to the sample in Section 3.6.2 shall be computed. We choose the formula given in Corollary 4.2 for the computation of $\lambda = \ell \cdot \sigma$ using Lemma 4.3. Therefore we need the following last exit generating functions for the computation of σ:

$$\begin{aligned}L_1(o_1, A|z) &= L_1(o_1, C|z) = L_1(o_1, E|z) = \frac{1}{5}z,\\ L_1(o_1, B|z) &= \frac{1}{5}z \cdot \frac{1}{2}z = \frac{1}{10}z^2,\\ L_1(o_1, D|z) &= \frac{1}{5}z + \frac{1}{5}z^2,\\ L_1(o_1, F|z) &= \frac{1}{5}z + \frac{1}{5}z \cdot \frac{1}{2}z = \frac{1}{5}z + \frac{1}{10}z^2,\\ L_2(o_2, G|z) &= L_2(o_2, H|z) = \frac{1}{2}z.\end{aligned}$$

In Section 3.6.2, we have obtained $\xi_1(1) \approx 0.66571$ and $\xi_2(1) = \xi_3(1) \approx 0.37231$, and also $\ell \approx 0.33089$. We get

$$\sigma \approx 1.02027$$

and thus

$$\lambda = \ell \cdot \sigma \approx 0.33760.$$

Furthermore,

$$\sigma_1 \approx 0.40591 \quad \text{and} \quad \sigma_2 = \sigma_3 \approx 0.30718$$

and thus

$$\lambda_1 = \sigma_1 \cdot \ell \approx 0.13431 \quad \text{and} \quad \lambda_2 = \lambda_3 = \sigma_2 \cdot \ell \approx 0.10164.$$

We extend this sample to another length function l on $X_1 * X_2 * X_3$ arising from the length functions l_1, l_2 and l_3 on X_1, X_2 and X_3 with

$$l_1(A) = 1,\ l_1(B) = 5,\ l_1(C) = 2,\ l_1(D) = 1,\ l_1(E) = 2,\ l_1(F) = 3,$$
$$l_2(G) = 1,\ l_2(H) = 2,\ l_3(I) = 3,\ l_3(J) = 3.$$

For instance, we have $S_1(2) = \{C, E\}$. We are interested in the rate of escape w.r.t. the associated length function l on the free product $X_1 * X_2 * X_3$. By Corollary 4.5, we obtain

$$\lim_{n\to\infty} \frac{1}{n} l(Z_n) \approx 0.70587.$$

4.4.2 Free Product with an Infinite Factor

Consider the infinite graph from Figure 4.1, denoted by X_1, with its transition probabilities and the Cayley graph X_2 of $\mathbb{Z}/3\mathbb{Z} = \langle b \mid b^3 = e_2 \rangle$, where e_2 is the identity on $\mathbb{Z}/3\mathbb{Z}$ and $\mu_2(b) = \mu_2(b^2) = 1/2$.

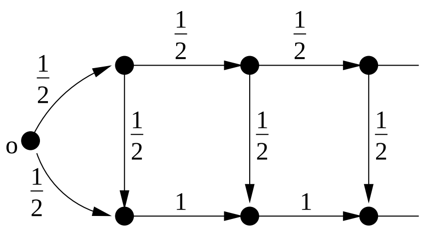
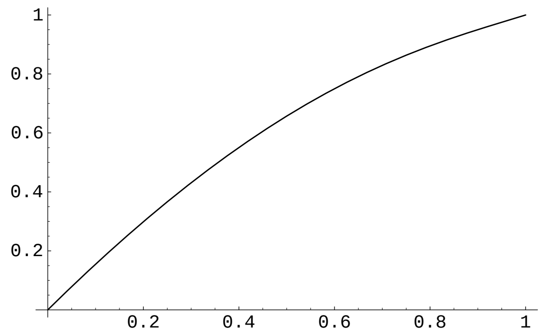


Figure 4.1: A simple infinite graph

Choose $0 < \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = 1$. Consider now the associated random walk on the free product $X = X_1 * X_2$, for which we want to compute the rate of escape w.r.t. the minimal path length. By Lemma 2.12, the associated Green function $G(o, o|z)$ has radius of convergence bigger than 1.



α_1	ℓ	λ
1/6	10/51	347/1377
1/5	8/35	23/77
1/4	3/11	137/374
1/3	1/3	17/36
1/2	2/5	23/35
2/3	8/21	152/189
3/4	1/3	19/22
4/5	16/55	128/143

Figure 4.2: Rate of escape for the random walk of Sample 4.4.2

We have obviously

$$G_1(o_1,o_1|z)=1,\ F_1(s,o_1|z)=0 \text{ for all } s\in\mathcal{S}(o_1) \text{ and } \bar{H}_2(z)=0.$$

From Section 3.6.1 we have

$$F_2(b,e_2|z)=F_2(b^2,e_2|z)=\frac{z}{2-z}\,,\ G_2(o_2,o_2|z)=\frac{2-z}{2-z-z^2},$$

and

$$\bar{H}_1(z)=\frac{\alpha_2\ z\ \xi_2(z)}{2-\xi_2(z)}.$$

Thus, we obtain

$$\xi_2(z)=\alpha_2\cdot z \quad \text{and} \quad \xi_1(z)=\frac{\alpha_1 z}{1-\frac{\alpha_2^2z^2}{2-\alpha_2 z}}.$$

Furthermore, we get

$$\begin{aligned}
&\sum_{n\geq 1}\sum_{y\in S_1(n)} n\cdot L_1(o_1,y|\xi_1)\\
=\ &\sum_{n\geq 1} n\cdot\left[\left(\frac{1}{2}\right)^n\xi_1^n+\frac{1}{2}\xi_1^n+\frac{1}{2}\left(\sum_{k=1}^{n}\frac{1}{2^k}\right)\xi_1^{n+1}\right]\\
=\ &\left(\frac{\xi_1}{2}-\frac{\xi_1^2}{4}\right)\cdot\frac{\partial}{\partial z}\left[\frac{1}{1-z}\right]\left(\frac{\xi_1}{2}\right)+\left(\frac{\xi_1}{2}+\frac{\xi_1^2}{2}\right)\cdot\frac{\partial}{\partial z}\left[\frac{1}{1-z}\right](\xi_1)\\
=\ &\frac{1}{2}\,\xi_1\,(4-3\xi_1+\xi_1^2)\,(1-\xi_1)^{-2}\,(2-\xi_1)^{-1}.
\end{aligned}$$

Moreover,

$$\sum_{n\geq 1}\sum_{y\in S_2(n)} n\cdot L_2(o_2,y|\xi_2) = \sum_{y\in\{b,b^2\}} L_2(o_2,y|\xi_2) = \frac{\frac{1}{1-\xi_2} - G_2(e_2,e_2\mid\xi_2)}{G_2(e_2,e_2\mid\xi_2)}.$$

Thus, we can compute σ by Lemma 4.3 in dependence of α_1 and $\alpha_2 = 1-\alpha_1$, and by that $\lambda = \sigma\cdot\ell$. Figure 4.2 shows the value of λ on the y-axis in dependence of α_1 on the x-axis.

4.5 Summary

We have presented formulas for the rate of escape w.r.t. different length functions. While the formula in Section 4.1 can only handle minimal path lengths, the formula presented in Section 4.2 can be applied to length functions arising from arbitrary bounded length functions on the graphs X_i. In Section 4.3, we extrapolated, similarily to Section 3.4, also a formula for the i-th partial rate of escape w.r.t. the minimal path length. Sample computations have shown applications of these formulas. In general, however, the practicability of the presented formulas is strictly related to the knowledge of the generating functions of the single factors X_i.

Part II

Acceleration of Lamplighter Random Walks

Chapter 5

Lamplighter Random Walks

Consider a transitive, connected, locally finite graph G equipped with a metric induced by weights on the edges and assume that there sits a lamp at each vertex, which can be on or off. A lamplighter moves randomly along the edges of the graph and may switch the lamps on or off. In this chapter, we will describe a suitable algebraic structure for this purpose and define transient random walks on it with different options for the lamplighter what to do in one step. Our aim in the upcoming Chapter 6 is to show that – under suitable assumptions – the lamplighter random walk escapes strictly faster to infinity than its projection onto G. Furthermore, in Chapter 7 we will give bounds for the rate of escape of the lamplighter random walk for the case that G is a homogeneous tree.

5.1 Lamplighter Graphs

Suppose we are given a transitive, connected, locally finite graph $\mathsf{G} = (V, E)$, called base graph, with root o. We allow only *symmetric* sets of edges E without loops and we write $x \sim y$ if $(x, y) \in E$. We assign a weight $w(x, y) = w(y, x) > 0$ to each edge $(x, y) \in E$. An automorphism $\gamma \in \mathrm{AUT}(\mathsf{G})$ of the weighted graph G is weight-preserving if $x_1 \sim x_2$ implies $w(x_1, x_2) = w(\gamma x_1, \gamma x_2)$ for all $x_1, x_2 \in V$. The set of weight-preserving automorphisms of G is denoted by $\mathrm{AUT}(\mathsf{G}, w)$, which is a subgroup of $\mathrm{AUT}(\mathsf{G})$. Analogously, the weighted graph G is called weight-transitive if for all $x, y \in V$ there is $\gamma \in \mathrm{AUT}(\mathsf{G}, w)$ such that $\gamma x = y$. From now on we assume that G is *weight-transitive*. We write

$$R_1 := \max\bigl\{w(o, x) \mid o \sim x\bigr\} \quad \text{and} \quad r_1 := \min\bigl\{w(o, x) \mid o \sim x\bigr\}.$$

The weight of a path $[x_0, x_1, \ldots, x_n]$ is $\sum_{i=1}^{n} w(x_{i-1}, x_i)$ (while its length is n). Moreover, a metric on G denoted by $d(x, y)$ for $x, y \in V$ is given by the minimal weight of all paths joining x and y. The ball $B(x,r)$ centered at x with radius $r \geq 0$ is given by the set of all vertices $x' \in V$ with $d(x,x') \leq r$.

We now show how to get a compactification of G. A one-way infinite path is an infinite sequence of vertices $[x_0, x_1, x_2, \dots]$ such that $x_i \sim x_{i+1}$ and $x_i \neq x_j$ for all $i,j \in \mathbb{N}_0$ with $i \neq j$. Two one-way infinite paths are equivalent if there is a third one-way infinite path which meets both paths infinitely often. An end of G is an equivalence class of one-way infinite paths. The set of ends is denoted by $\partial\mathsf{G}$. For finite $F \subseteq V$, denote by $\mathsf{G} \setminus F$ the graph with vertex set $V \setminus F$ and set of edges $E \cap (V \setminus F)^2$. By local finiteness, each one-way infinite path has all but finitely many vertices in one connected component of $\mathsf{G} \setminus F$. If $\omega \in \partial\mathsf{G}$, then for each $k \in \mathbb{N}_0$ there is exactly one connected component $C_k(\omega)$ of $\mathsf{G} \setminus B(o,k)$ such that all one-way infinite paths in ω end up in $C_k(\omega)$. The completion $\mathsf{G} \cup \partial\mathsf{G}$ becomes a compact space with the discrete topology on G, while a neighbourhood basis of $\omega \in \partial\mathsf{G}$ is given by the sets $C_k(\omega)$.

Assume now that there sits a *lamp* at each vertex of G, which can be switched off or on, encoded by elements of $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$: '0' represents the state 'off' and '1' the state 'on'. We think of a lamplighter starting at o with all lamps off, walking along G and switching lamps on and off. To describe the configuration of the lamp states we use functions $\eta : V \to \mathbb{Z}_2$ with finite support. Moreover, the set of finitely supported configurations of lamps is given by

$$\mathcal{N} := \big\{ \eta : V \to \mathbb{Z}_2 \;\big|\; \operatorname{card} \operatorname{supp}(\eta) < \infty \big\}.$$

Denote by $\mathbf{0}$ the zero function and by $\mathbb{1}_x$ the indicator function w.r.t. $x \in V$, that is, $\mathbb{1}_x(x) = 1$ and $\mathbb{1}_x(y) = 0$ for $y \in V \setminus \{x\}$. We now consider the Lamplighter Graph $\mathbb{Z}_2 \wr \mathsf{G}$ with vertex set $\mathcal{N} \times V$ and the adjacency relation given by

$$(\eta, x) \sim_{\mathcal{L}} (\eta', x') \quad \Longleftrightarrow \quad \begin{cases} x \sim x' \text{ and } \eta = \eta', & \text{or} \\ x = x' \text{ and } \eta' = \eta \oplus \mathbb{1}_x \end{cases}$$

for all $(\eta,x), (\eta',x') \in \mathcal{N} \times V$, where $\oplus$ is the componentwise addition modulo 2. The graph $\mathbb{Z}_2 \wr \mathsf{G}$ is again transitive; see Woess [44, Proposition 1.1]. We lift $d(\cdot,\cdot)$ to a (pseudo-) metric $d_{\mathbb{Z}_2 \wr \mathsf{G}}(\cdot,\cdot)$ on $\mathbb{Z}_2 \wr \mathsf{G}$ by assigning the following weights to the edges of $\mathbb{Z}_2 \wr \mathsf{G}$: if $(\eta,x) \sim_{\mathcal{L}} (\eta',x')$ with $x \neq x'$, then the corresponding edge has weight $w(x,x')$; we assign to the remaining edges the arbitrary, but fixed value $\delta_{\mathcal{L}} \geq 0$, that is, $d_{\mathbb{Z}_2 \wr \mathsf{G}}\big((\eta,x),(\eta \oplus \mathbb{1}_x, x)\big) = \delta_{\mathcal{L}}$ for all $(\eta,x) \in \mathcal{N} \times V$. The distance of (η,x) and (η',x') is then the minimal weight of all paths in $\mathbb{Z}_2 \wr \mathsf{G}$ joining both elements. We write

$$|(\eta,x)| := d_{\mathbb{Z}_2 \wr \mathsf{G}}\big((\mathbf{0},o),(\eta,x)\big) = \widehat{d}(\eta,x) + \delta_{\mathcal{L}} \cdot \operatorname{card}\operatorname{supp}(\eta),$$

where $\widehat{d}(\eta,x)$ is the weight of an optimal 'travelling salesman' tour on G from o to x with visiting each point in $\operatorname{supp}(\eta)$. Obviously, it is $d(o,x) \leq |(\eta,x)|$.

Note that $d_{\mathbb{Z}_2 \wr \mathsf{G}}(\cdot,\cdot)$ is only a pseudo-metric if $\delta_{\mathcal{L}} = 0$. We will show in the next section that $\mathbb{Z}_2 \wr \mathsf{G}$ is also weight-transitive.

5.2 Random Walks on Lamplighter Graphs

In this section we show how to lift a random walk on G in a natural way to a random walk on $\mathbb{Z}_2 \wr \mathsf{G}$. For this purpose, we consider a *transient* random walk with *bounded range* on G, called base random walk, starting at o and governed by a transition matrix P_0, where single step and n-step transition probabilities are denoted by $p_0(x,y)$ and $p_0^{(n)}(x,y)$ for $x,y \in V$. We make again the basic assumption that $x \sim y$ implies $p_0(x,y) > 0$. To obtain a *space-homogeneous* Markov chain not only with respect to the adjacency relation but also with respect to the weights of the edges, we assume that there is a subgroup Γ of $\mathrm{AUT}(\mathsf{G}, w)$ acting weight-transitively on G such that $x \sim y$ implies $p_0(x,y) = p_0(\gamma x, \gamma y)$ for all $\gamma \in \Gamma$. As P_0 has bounded range, we may add edges between non-neighbours x and y with weight $d(x,y)$, if $p_0(x,y) > 0$, and obtain a new weight-transitive graph, on which we may consider the random walk. Thus, we may assume w.l.o.g. that for $x \neq y$ we have $p_0(x,y) > 0$ if and only if $x \sim y$. By space-homogeneity and local finiteness, there is $\varepsilon_0 > 0$ such that $p_0(x,y) \geq \varepsilon_0$ for all neighbours $x,y \in V$.

In order to lift P_0 to a suitable random walk on $\mathbb{Z}_2 \wr \mathsf{G}$, we introduce a family of discrete probability measures $(\mu_x)_{x \in V}$ on $\mathcal{N}$ satisfying the following conditions:

(i) There is a number $R_2 \geq 0$ such that $\mu_x(\eta) > 0$ implies $d(x,y) \leq R_2$ for all $y \in \mathrm{supp}(\eta)$, that is, each μ_x has finite support.

(ii) The space homogeneity property holds, that is, for each $\gamma \in \Gamma$ we have $\mu_{\gamma x}(\gamma \eta) = \mu_x(\eta)$, where $(\gamma\eta)(y) := \eta(\gamma^{-1} y)$ for $y \in V$.

(iii) We assume $\zeta := \mu_x(\mathbf{0}) > 0$ for all $x \in V$.

(iv) For sake of simplicity, we assume that for each $x \in V$ there is some $\eta_x \in \mathcal{N}$ with $\eta_x(x) = 1$ and $\mu_x(\eta_x) > 0$.

The measures μ_x describe laws for switching lamps in some bounded neighbourhood of x when standing at x. Condition (iv) is no real restriction: assuming that each lamp can be switched on and off by the lamplighter with positive probability, one can fix for each vertex $x \in V$ another vertex $y \in V$ at bounded distance from x such that the lamp at x can be switched with positive probability when standing at y. To avoid constructing detours to y, if we want to switch the lamp at x, we assume that condition (iv) holds, although it is not necessary for our results.

Actually, it is sufficient to define μ_o such that for each $\gamma \in \Gamma$ with $\gamma o = o$ the equation $\mu_o(\gamma\eta) = \mu_o(\eta)$ holds. As Γ acts transitively on G, the measures μ_x can be retrieved by $\mu_x(\gamma\eta) = \mu_o(\eta)$ for $\gamma \in \Gamma$ with $\gamma o = x$. As μ_o has finite support, there is some $\xi_0 > 0$ such that $\mu_x(\eta) \geq \xi_0$ for all $x \in V$ and all $\eta \in \mathrm{supp}(\mu_x)$. For instance, if $\mu_x(\mathbb{1}_x) = p$ and $\mu_x(\mathbf{0}) = 1-p$ for any $p \in (0,1)$, then the requested conditions hold.

Now we can define a lamplighter random walk P on $\mathbb{Z}_2 \wr \mathsf{G}$ by the following transition probabilities:

$$p\big((\eta,x),(\eta',x')\big) := \sum_{\substack{\eta_1,\eta_2\in\mathcal{N},\\ \eta_1\oplus\eta_2=\eta\oplus\eta'}} \mu_x(\eta_1)\cdot p_0(x,x')\cdot \mu_{x'}(\eta_2).$$

This random walk corresponds to the model ('Switch-Walk-Switch'), where – in one step – the lamplighter *may* flip some lamp states in the neighbourhood of his actual position x (according to the probability measure μ_x), then walks to a random vertex x' and *may* flip the lamp states in a neighbourhood of x' (according to $\mu_{x'}$). The corresponding n-step transition probabilities are denoted by $p^{(n)}(\cdot,\cdot)$. The lamplighter random walk is described by the sequence of random variables $(Z_n)_{n\in\mathbb{N}_0}$. More precisely, we write $Z_n = \big(\eta_n, X_n\big)$, where η_n is the random configuration at time n and X_n is the random vertex where the lamplighter stands at time n. Initially, $Z_0 = (\mathbf{0}, o)$. We denote by $\mathbb{P}_{(\eta,x)}[\,\cdot\,]$ the probability measure that governs the random walk starting at (η,x) instead of $(\mathbf{0},o)$.

Proposition 5.1. *There is a subgroup Γ' of* $\mathrm{AUT}(\mathbb{Z}_2 \wr \mathsf{G})$ *acting transitively on $\mathbb{Z}_2 \wr \mathsf{G}$ such that for all $\underline{\gamma} \in \Gamma'$, $(\eta,x),(\eta',x') \in \mathcal{N}\times V$*

$$d_{\mathbb{Z}_2\wr\mathsf{G}}\big(\underline{\gamma}(\eta,x),\underline{\gamma}(\eta',x')\big) = d_{\mathbb{Z}_2\wr\mathsf{G}}\big((\eta,x),(\eta',x')\big)$$

and

$$p\big(\underline{\gamma}(\eta,x),\underline{\gamma}(\eta',x')\big) = p\big((\eta,x),(\eta',x')\big).$$

Proof. We follow Woess [44, Section 1.2] to construct Γ'. Define

$$\Phi := \big\{\mathbb{1}_A \mid A \subseteq V \text{ finite}\big\},$$

where $\mathbb{1}_A$ is the indicator function w.r.t. the set A, that is, $\mathbb{1}_A(x) = 1$, if $x \in A$, and $\mathbb{1}_A(x) = 0$ otherwise. Then Φ becomes a group when equipped with the pointwise addition modulo 2. For $\mathbb{1}_A \in \Phi$, $\gamma \in \Gamma$ and $(\eta,x) \in \mathcal{N}\times V$ we define an action of $(\mathbb{1}_A,\gamma)$ on $\mathcal{N}\times V$ by

$$(\mathbb{1}_A,\gamma)(\eta,x) := (\mathbb{1}_A \oplus (\gamma\eta), \gamma x),$$

where $(\gamma\eta)(y) := \eta(\gamma^{-1}y)$ for $y \in V$. Moreover, with $\mathbb{1}_B \in \Phi$ and $\gamma' \in \Gamma$ the composition is defined as

$$\big(\mathbb{1}_A,\gamma\big)\big(\mathbb{1}_B,\gamma'\big) := \big(\mathbb{1}_A \oplus (\gamma\mathbb{1}_B), \gamma\gamma'\big).$$

With this operation we obtain the semi-direct product $\Gamma' := \Phi \rtimes \Gamma$, which is a subgroup of $\mathrm{AUT}(\mathbb{Z}_2 \wr \mathsf{G})$ and acts transitively on $\mathbb{Z}_2 \wr \mathsf{G}$; compare with Woess [44, Proposition 1.1(a)]. We now show that each automorphism of $\Phi \rtimes \Gamma$ maps edges to edges with the same weight, from which the first equation of the proposition follows directly. If $(\mathbb{1}_A, \gamma) \in \Phi \rtimes \Gamma$ and $\eta \in \mathcal{N}, x, y \in V$ with $x \sim y$, then

$$\begin{aligned} & d\big((\mathbb{1}_A,\gamma)(\eta,x),(\mathbb{1}_A,\gamma)(\eta,y)\big) \\ = \;& d\big((\mathbb{1}_A \oplus (\gamma\eta),\gamma x),(\mathbb{1}_A \oplus (\gamma\eta),\gamma y)\big) \\ = \;& d\big((\eta,x),(\eta,y)\big). \end{aligned}$$

Furthermore:

$$\begin{aligned} & d\big((\mathbb{1}_A,\gamma)(\eta,x),(\mathbb{1}_A,\gamma)(\eta\oplus\mathbb{1}_x,x)\big) \\ = \;& d\big((\mathbb{1}_A \oplus (\gamma\eta),\gamma x),(\mathbb{1}_A \oplus (\gamma(\eta\oplus\mathbb{1}_x)),\gamma x)\big) \\ = \;& d\big((\mathbb{1}_A \oplus (\gamma\eta),\gamma x),(\mathbb{1}_A \oplus (\gamma\eta)\oplus\mathbb{1}_{\gamma x},\gamma x)\big) \\ = \;& \delta_{\mathcal{L}} = d\big((\eta,x),(\eta\oplus\mathbb{1}_x,x)\big). \end{aligned}$$

Now we prove the second of the proposed equations. Let $(\eta', x') \in \mathcal{N} \times V$. For $\eta_1, \eta_2 \in \mathcal{N}$ with $\eta_1 \oplus \eta_2 = \eta \oplus \eta'$ and $v \in V$ we have modulo 2

$$\begin{aligned} (\gamma\eta_1)(v) + (\gamma\eta_2)(v) &= \eta_1(\gamma^{-1}v) + \eta_2(\gamma^{-1}v) \\ &= \eta(\gamma^{-1}v) + \eta'(\gamma^{-1}v) \\ &= (\gamma\eta)(v) + (\gamma\eta')(v). \end{aligned}$$

By transitivity of P_0 and space homogeneity of the μ_x, we can conlude:

$$\begin{aligned} & p\big((\mathbb{1}_A,\gamma)(\eta,x),(\mathbb{1}_A,\gamma)(\eta',x')\big) \\ = \;& \sum_{\substack{\eta_1,\eta_2\in\mathcal{N}, \\ \gamma\eta_1\oplus\gamma\eta_2=(\gamma\eta)\oplus(\gamma\eta')}} \mu_{\gamma x}(\gamma\eta_1)\cdot p_0(\gamma x,\gamma x')\cdot \mu_{\gamma x'}(\gamma\eta_2) \\ = \;& \sum_{\substack{\eta_1,\eta_2\in\mathcal{N}, \\ \eta_1\oplus\eta_2=\eta\oplus\eta'}} \mu_x(\eta_1)\cdot p_0(x,x')\cdot \mu_{x'}(\eta_2) \\ = \;& p\big((\eta,x),(\eta',x')\big). \end{aligned}$$

□

Our random walk projects onto the two processes X_n on the graph G and η_n on $\mathcal{N}$, of which we can investigate convergence. Observe that by transience each finite subset of V is visited only finitely often yielding that $(\eta_n)_{n\in\mathbb{N}_0}$ converges pointwise to a random limit configuration $\eta_\infty : V \to \mathbb{Z}_2$, which is not necessarily finitely supported. On the other hand, X_n converges to a random end in $\partial\mathsf{G}$; in particular, $d(o, X_n)$ goes to infinity.

By weight-transitivity of G and Proposition 5.1, it follows as an consequence from Theorem 1.3 that there are constants $\ell_0, \ell \in \mathbb{R}_{\geq}$ such that

$$\ell_0 = \lim_{n\to\infty} \frac{d(o, X_n)}{n} \quad \text{almost surely}$$

and

$$\ell = \lim_{n\to\infty} \frac{|Z_n|}{n} \quad \text{almost surely.}$$

The number ℓ_0 is the rate of escape or drift of the lamplighter random walk's projection onto G and ℓ is the rate of escape of the lamplighter random walk. Our aim is to show that, under suitable assumptions on G, ℓ is strictly bigger than ℓ_0, that is, the lamplighter random walk escapes faster to infinity than its projection onto G. In general, the acceleration of the lamplighter random walk is not obvious. Let us briefly mention an example for the case where $\ell = \ell_0$:

Example 5.2: Consider the Cayley graph of $\mathbb{Z}$ w.r.t. the generators ± 1 equipped with a transient random walk, that is, $p_0(z, z+1) = q \in (1/2, 1]$, $p_0(z, z-1) = 1 - q$ for all $z \in \mathbb{Z}$. We consider the case $\delta_{\mathcal{L}} = 0$ and we set $\mu_z(\mathbb{1}_z) = p \in (0;1)$ and $\mu_z(\mathbf{0}) = 1 - p$. For $\eta : \mathbb{Z} \to \mathbb{Z}_2$, we write

$$\eta^+(z) := \begin{cases} \eta(z), & \text{if } z \geq 0 \\ 0, & \text{if } z < 0 \end{cases} \quad \text{and} \quad \eta^-(z) := \begin{cases} \eta(z), & \text{if } z < 0 \\ 0, & \text{if } z \geq 0 \end{cases}.$$

Then $\eta = \eta^+ \oplus \eta^-$ and, after the last visit to 0, $|Z_n| = |(\eta_n^-, 0)| + |(\eta_n^+, X_n)|$. As $|(\eta_n^+, X_n)| = d(0, X_n)$ and $|(\eta_n^-, 0)|$ remains constant for large n, it follows that $\ell = \ell_0$. Compare also with Bertacchi [1].

Chapter 6

Acceleration of Lamplighter Random Walks

In this chapter we will show that, under suitable assumptions, the lamplighter random walk escapes faster to infinity than its projection onto the base graph G, that is, we want to show $\ell > \ell_0$. For this purpose, we have to distinguish if $\delta_{\mathcal{L}} = 0$, $\ell_0 > 0$ respectively, or not. Assuming $\ell_0 > 0$, the case $\delta_{\mathcal{L}} > 0$ is discussed in the following section, while the case $\delta_{\mathcal{L}} = 0$ is investigated in Section 6.2 for graphs with infinitely many ends (Section 6.2.1) and for graphs with exactly two ends (Section 6.2.2). The remaining case $\ell_0 = 0$ is discussed in Section 6.3, where the proposed inequality follows from results of Kaimanovich and Vershik [15] and of Varopoulos [38]. Finally, we give in Section 6.4 some additional remarks regarding extensions of the presented results.

6.1 The Case $\delta_{\mathcal{L}} > 0$

In this section we consider the case that edges in $\mathbb{Z}_2 \wr \mathsf{G}$ corresponding to neighbours $(\eta, x) \sim_{\mathcal{L}} (\eta \oplus \mathbb{1}_x, x)$ have weight $\delta_{\mathcal{L}} > 0$. Furthermore, we assume $\ell_0 > 0$ for the rest of this section. We want to prove $\ell > \ell_0$.

Setting $R := R_1 + R_2$ we define for $k \in \mathbb{N}_0$ the exit times

$$\mathbf{e}_k := \min\big\{m \in \mathbb{N} \;\big|\; \forall n \geq m : d(o, X_n) \geq kR\big\}.$$

By transience, $\mathbf{e}_k < \infty$ holds almost surely for all $k \in \mathbb{N}_0$. Observe that we have $d(o, X_{\mathbf{e}_k}) < kR + R_1$. With this notion we obtain:

Lemma 6.1.

$$\ell_0 = \lim_{k \to \infty} \frac{kR}{\mathbf{e}_k} \qquad \mathbb{P}_o - a.s.$$

Proof. We have almost surely

$$\ell_0 = \lim_{k\to\infty} \frac{d(o, X_{\mathbf{e}_k})}{\mathbf{e}_k} = \lim_{k\to\infty} \frac{d(o, X_{\mathbf{e}_k})}{kR}\frac{kR}{\mathbf{e}_k}.$$

Define the random variable $w_k := d(o, X_{\mathbf{e}_k}) - kR \geq 0$. As the random walk has bounded range, w_k is bounded by R_1. Thus w_k/k tends to zero, which yields the claim. □

The last lemma implies

$$\ell = \lim_{k\to\infty} \frac{|Z_{\mathbf{e}_k}|}{\mathbf{e}_k} = \lim_{k\to\infty} \frac{|Z_{\mathbf{e}_k}|}{kR}\frac{kR}{\mathbf{e}_k} = \ell_0 \cdot \lim_{k\to\infty} \frac{|Z_{\mathbf{e}_k}|}{kR}. \tag{6.1}$$

As $\ell_0 > 0$, the limit $\ell_1 := \lim_{k\to\infty} |Z_{\mathbf{e}_k}|/(kR)$ exists almost surely and is almost surely constant. We will show that this limit is strictly bigger than 1, which will yield the proposed inequality $\ell > \ell_0$.

We introduce some further notation. For $k \in \mathbb{N}_0$, let be

$$S_k := \{x \in V \mid kR \leq d(o, x) < kR + R_1\}$$

the R_1-annulus at distance kR in G and

$$\mathbf{s}_k := \min\{n \in \mathbb{N} \mid X_n \in S_k\}$$

the hitting time of S_k. By transience, $\mathbf{s}_k < \infty$ almost surely. For $k \in \mathbb{N}_0$, let the pseudo-increments be

$$\Delta_k := \begin{cases} \delta_{\mathcal{L}}, & \text{if } \eta_\infty(X_{\mathbf{s}_k}) = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Observe that $\eta_n(X_{\mathbf{s}_j})$ remains constant for all $n \geq \mathbf{e}_k$ and all $j < k$. Thus, we have for all $k \in \mathbb{N}$

$$|Z_{\mathbf{e}_k}| \geq kR + \sum_{j=0}^{k-1} \Delta_j,$$

or equivalently,

$$\frac{|Z_{\mathbf{e}_k}|}{kR} \geq 1 + \frac{1}{kR}\sum_{j=0}^{k-1} \Delta_j. \tag{6.2}$$

We will need the following lemma several times in the sequel:

Lemma 6.2. *Let $s \in \mathbb{R}_+$ and $x, y \in V$ with $d(x, y) \leq s$. Then there is a path in G from x to y of length at most $\lfloor s/r_1 \rfloor$.*

Proof. Consider a path $[x = x_0, x_1, \ldots, y = x_m]$ inside G of minimal weight from x to y. Hence,

$$s \geq d(x, y) = \sum_{i=1}^{m} w(x_{i-1}, x_i) \geq m \cdot r_1.$$

Dividing by r_1 both sides of the inequality yields the lemma. $\square$

Now we want to state a lower bound for the probability $\mathbb{P}_{(\eta,x)}[\eta_\infty(x) = 1]$ for $(\eta, x) \in \mathcal{N} \times V$ with $\eta(x) = 1$.

Lemma 6.3. *There are $\kappa \in \mathbb{N}$ and $\tilde{p} > 0$ such that for all $(\eta, x) \in \mathcal{N} \times V$ with $\eta(x) = 1$*

$$\mathbb{P}_{(\eta,x)}[\eta_\infty(x) = 1] \geq \left(\varepsilon_0\, \zeta^2\right)^\kappa \cdot \tilde{p} > 0.$$

Proof. Let $(\eta, x) \in \mathcal{N} \times V$ with $\eta(x) = 1$. By transience and bounded range of the random walk P_0, there is at least one vertex $y \in V$ satisfying $R_2 < d(x, y) \leq R_1 + R_2$ such that

$$\tilde{p} := \mathbb{P}_{(\eta,y)}[\forall n \geq 1 : X_n \notin B(x, R_2)] > 0.$$

Moreover, there is a path from x to y inside the graph G of length at most $\kappa := \lfloor (R_1 + R_2)/r_1 \rfloor$. Thus, the probability of walking from (η, x) to (η, y) with no lamp switches during this walk is at least $\left(\varepsilon_0\, \zeta^2\right)^\kappa$. By transitivity, the choice of $\tilde{p}$ and κ is independent of (η, x). Thus follows the claim. $\square$

Recall that $\mu_x(\eta) \geq \xi_0$ for all $x \in V$ and $\eta \in \operatorname{supp} \mu_x$. By changing the lamp state at x, when leaving x for the first time, the last lemma also yields

$$\mathbb{P}_{(\eta,x)}[\eta_\infty(x) = 1] \geq \frac{\xi_0}{\zeta} \cdot \left(\varepsilon_0\, \zeta^2\right)^\kappa \cdot \tilde{p}$$

for $(\eta, x) \in \mathcal{N} \times V$ with $\eta(x) = 0$. The next lemma gives a non-trivial uniform lower bound for $\mathbb{E}[\Delta_k]$:

Lemma 6.4. *There is $B_1 > 0$ such that $\mathbb{E}[\Delta_k] \geq B_1$ for all $k \in \mathbb{N}_0$.*

Proof. In order to bound $\mathbb{P}[\Delta_k = \delta_{\mathcal{L}}]$ uniformly from below, we decompose according to all possible states of $X_{\mathbf{s}_k}$, where the lamp – if necessary – will be switched on, followed by a walk with no lamp switches to some vertex $y \in V \setminus B(X_{\mathbf{s}_k}, R_2)$, from which the random walk does not return to $B(X_{\mathbf{s}_k}, R_2)$.

By vertex-transitivity and Lemma 6.3, the probability of starting in $X_{\mathbf{s}_k}$ with $\eta_{\mathbf{s}_k}(X_{\mathbf{s}_k}) = s \in \mathbb{Z}_2$ and walking to some vertex $y \in V \setminus B(X_{\mathbf{s}_k}, R_2)$ with no lamp switches until reaching y and no visit in $B(X_{\mathbf{s}_k}, R_2)$ after reaching y is at least $(\xi_0/\zeta)^{1-s} \cdot \left(\varepsilon_0\, \zeta^2\right)^\kappa \cdot \tilde{p}$.

Observe that we have by transience:

$$\sum_{x\in S_k}\sum_{n\geq 0}\mathbb{P}[\mathbf{s}_k = n, X_n = x] = \mathbb{P}[\mathbf{s}_k < \infty] = 1.$$

We get:

$$\begin{aligned}
&\mathbb{P}[\Delta_k = \delta_{\mathcal{L}}]\\
= \quad & \sum_{x\in S_k}\sum_{n\geq 0}\sum_{\substack{x_1,\dots,x_{n-1}\in V,\\ d(o,x_j)<kR}}\sum_{i\in\{0,1\}}\mathbb{P}\left[\begin{array}{c} X_1 = x_1,\dots,X_{n-1}=x_{n-1},\\ X_n = x, \eta_n(x) = i, \eta_\infty(x) = 1\end{array}\right]\\
\geq \quad & \sum_{x\in S_k}\sum_{n\geq 0}\sum_{\substack{x_1,\dots,x_{n-1}\in V,\\ d(o,x_j)<kR}}\mathbb{P}\left[\begin{array}{c} X_1 = x_1,\dots,X_{n-1}=x_{n-1},\\ X_n = x, \eta_n(x) = 1\end{array}\right]\cdot\big(\varepsilon_0\,\zeta^2\big)^{\kappa}\cdot\tilde{p}\\
& +\mathbb{P}\left[\begin{array}{c} X_1 = x_1,\dots,X_{n-1}=x_{n-1},\\ X_n = x, \eta_n(x) = 0\end{array}\right]\cdot\frac{\xi_0}{\zeta}\cdot\big(\varepsilon_0\,\zeta^2\big)^{\kappa}\cdot\tilde{p}\\
\geq \quad & \sum_{x\in S_k}\sum_{n\geq 0}\mathbb{P}[\mathbf{s}_k = n, X_n = x]\cdot\min\{\xi_0/\zeta, 1\}\cdot\big(\varepsilon_0\,\zeta^2\big)^{\kappa}\cdot\tilde{p}\\
= \quad & \xi_0\cdot\varepsilon_0^{\kappa}\cdot\zeta^{2\kappa-1}\cdot\tilde{p} =: b_1.
\end{aligned}\tag{6.3}$$

We obviously have $b_1 > 0$, and with $B_1 := \delta_{\mathcal{L}}\cdot b_1$ we get the inequality $\mathbb{E}[\Delta_k] \geq B_1 > 0$. □

Before we can prove the inequality $\ell_1 > 1$, we need the following lemma:

Lemma 6.5. *Suppose we are given a sequence of bounded, real-valued, non-negative random variables $(A_n)_{n\in\mathbb{N}_0}$. For $k\in\mathbb{N}$, let*

$$D_k := \frac{1}{k}\sum_{j=0}^{k-1}A_j.$$

Then

$$\mathbb{E}\big[\limsup_{k\in\mathbb{N}} D_k\big] \geq \limsup_{k\in\mathbb{N}}\mathbb{E}[D_k].$$

Proof. We have $0\leq A_n\leq c$ for some $c\in\mathbb{R}_{\geq}$ and all $n\in\mathbb{N}_0$, and consequently $0\leq D_k\leq c$ for all $k\in\mathbb{N}$. As

$$\limsup_{k\in\mathbb{N}} D_k = c - \liminf_{k\in\mathbb{N}}\big(c - D_k\big),$$

we can apply Fatou's Lemma and obtain

$$\begin{aligned}
\mathbb{E}\big[\limsup_{k\in\mathbb{N}} D_k\big] &= c - \int\liminf_{k\in\mathbb{N}}\big(c-D_k\big)\,d\mathbb{P}\\
&\geq c - \liminf_{k\in\mathbb{N}}\int\big(c-D_k\big)\,d\mathbb{P} = \limsup_{k\in\mathbb{N}}\mathbb{E}[D_k].
\end{aligned}$$

□

Now we can conclude:

Theorem 6.6. *For the lamplighter random walk with respect to the transitive, connected, locally finite base graph* G *and its metric* $d(\cdot,\cdot)$*, assuming* $\delta_{\mathcal{L}} > 0$*,*

$$\ell \geq \Big(1 + \frac{\delta_{\mathcal{L}}\, b_1}{R}\Big) \cdot \ell_0 > \ell_0,$$

where $R = R_1 + R_2$ *and* b_1 *is given by equation (6.3).*

Proof. By Lemma 6.4, we have the inequality

$$\mathbb{E}\Big[\sum_{j=0}^{k-1} \Delta_j/(kR)\Big] \geq \delta_{\mathcal{L}}\, b_1/R > 0.$$

Choose $A_n := \Delta_n/R$ and apply Lemma 6.5: by equation (6.2), we get $\ell_1 \geq 1 + \limsup_{k\in\mathbb{N}} D_k$, providing

$$\ell_1 \geq 1 + \mathbb{E}[\limsup_{k\in\mathbb{N}} D_k] \geq 1 + \limsup_{k\in\mathbb{N}} \mathbb{E}[D_k] \geq 1 + \frac{\delta_{\mathcal{L}}\, b_1}{R} > 1.$$

The rest follows from equation (6.1). □

6.2 The Case $\delta_{\mathcal{L}} = 0$

We distinguish if G has either exactly two or infinitely many ends. Recall that – in the transitive case – having three or more ends is equivalent to having infinitely many ends. For the rest of this section we assume $\ell_0 > 0$. For graphs with exactly one end and $\ell_0 > 0$, the question, when $\ell > \ell_0$ holds, remains open.

6.2.1 Graphs with Infinitely many Ends

In this section we want to prove $\ell > \ell_0$ in the – more delicate – case $\delta_{\mathcal{L}} = 0$ under the assumptions that $\ell_0 > 0$ and G has *infinitely* many ends. To prove the claim we adjust the considerations of the previous section.

As G is transitive and has infinitely many ends, there is some $r \in \mathbb{N}_0$ such that for every $x \in V$ the graph $\mathsf{G} \setminus B(x,r)$ consists of at least three different, infinite, connected components. For $n \in \mathbb{N}_0$, denote by $\mathcal{C}(x,n)$ the set of infinite, connected components of $\mathsf{G} \setminus B(x,n)$. We identify these components with their sets of vertices. By transience, there is at least one $C \in \mathcal{C}(o,r)$ such that the base random walk travels into C and returns to the ball $B(o,r)$ with a probability strictly smaller than 1. Considering the base random walk, we denote by $F\big(x, B(o,r)\big)$ the probability of starting at $x \in V$ and visiting after finite time an element of the set $B(o,r)$. Define

$$\bar{F} := \max\left\{ F\big(y, B(o,r)\big) \;\middle|\; \begin{array}{c} y \in V, r < d(o,y) \leq r + R_1, \\ F\big(y, B(o,r)\big) < 1 \end{array} \right\}.$$

Observe that $\bar{F} < 1$ and that for all $C \in \mathcal{C}(o,r)$ and all $y_1, y_2 \in C$ it is $F(y_1, B(o,r)) < 1$ if and only if $F(y_2, B(o,r)) < 1$. We now distinguish whether

$$\Lambda := \mathrm{card}\big\{ C \in \mathcal{C}(o,r) \;\big|\; \text{there is } x \in C \text{ with } F(x, B(o,r)) < 1 \big\} \geq 2$$

or not. In other words, we distinguish whether the projection of the random walk onto G converges to a deterministic end ($\Lambda = 1$) or not ($\Lambda \geq 2$).

6.2.1.1 Case $\Lambda \geq 2$

Define the function $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ by $\psi(n) = n(2r + R_1 + 1)$ and also for $k \in \mathbb{N}_0$ the exit times

$$\mathbf{e}_k := \min\big\{ m \in \mathbb{N} \;\big|\; \forall n \geq m : d(o, X_n) \geq \psi(k) \big\}, \tag{6.4}$$

the midpoint times

$$\mathbf{m}_k := \min\big\{ m \in \mathbb{N} \;\big|\; m \geq \mathbf{e}_k, d(o, X_m) \geq \psi(k) + r \big\}$$

and $\mathbf{M}_k := X_{\mathbf{m}_k}$. By transience, $\mathbf{e}_k, \mathbf{m}_k < \infty$ hold almost surely. Observe that $B(\mathbf{M}_k, r) \cap B(\mathbf{M}_{k-1}, r) = \varnothing$ and $X_{\mathbf{e}_{k+1}} \notin B(\mathbf{M}_k, r)$. See Figure 6.1. The

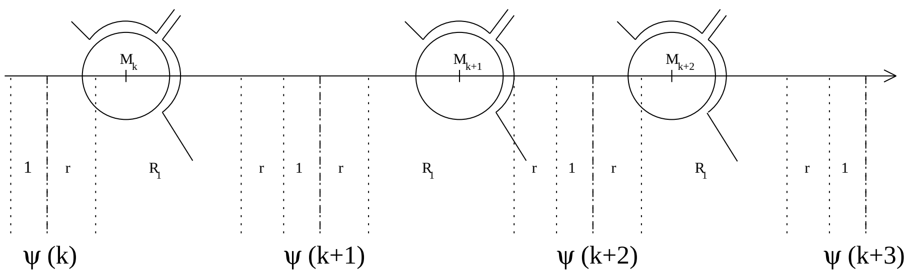


Figure 6.1: Exit and midpoint times

idea is to 'construct' non-intersecting balls between consecutive exit points $X_{\mathbf{e}_k}$ and $X_{\mathbf{e}_{k+1}}$, from which we create deviations yielding larger distances with positive probability. Analogously to the previous section we obtain

$$\ell_0 = \lim_{k\to\infty} \frac{\psi(k)}{\mathbf{e}_k} \quad \text{and} \quad \ell = \ell_0 \cdot \lim_{k\to\infty} \frac{|Z_{\mathbf{e}_k}|}{\psi(k)}. \tag{6.5}$$

As $\ell_0 > 0$, the limit $\ell_1 := \lim_{k\to\infty} |Z_{\mathbf{e}_k}|/\psi(k)$ exists almost surely and is almost surely constant. We show that this limit is strictly bigger than 1, yielding $\ell > \ell_0$. For this purpose, we proceed by bounding $|Z_{\mathbf{e}_k}|$ from below.

We start with some simple graph theoretical properties: if $w, x, y \in V$ such that $o, w \notin B(y, r)$ and x satisfying $d(y, x) \leq 2r + R_2 + 1$ is not in one of the components of $\mathcal{C}(y, r)$ containing o or w, then for all $y_1, y_2 \in B(y, r)$ we have

$$d(o, w) \leq d(o, y_1) + d(y_1, x) + d(x, y_2) + d(y_2, w). \tag{6.6}$$

See Figure 6.2. The distances $d(y_1, x)$ and $d(x, y_2)$ are bounded from below

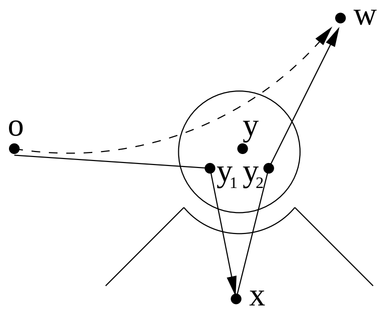


Figure 6.2: Deviation into a component of $\mathcal{C}(y, r)$

by $r + 1$. If a minimal-weighted path $[o, w_1, \dots, w]$ in G from o to w crosses the ball $B(y, r)$, where w_s is the first vertex of the path in $B(y, r)$ and w_t the last vertex in $B(y, r)$, then $d(w_s, w_t) \leq 2r$. Due to this inequality and relation (6.6) a minimal-weighted tour from o to w with visit at x before reaching w has a weight at least $d(o, w) + 2$. It is even not possible to switch the lamp at x, when walking on a minimal-weighted tour from o to w. To get a lower bound for $|Z_n|$ with $n \geq \mathbf{e}_k$, we are now interested in counting these '+2' increments, where Z_n plays the role of w and the $\mathbf{M}_j$'s, $j < k$, the role of y. For this purpose, we introduce further notation: we write

$$\widehat{\mathcal{C}}(\mathbf{M}_k, r) := \big\{ C \in \mathcal{C}(\mathbf{M}_k, r) \;\big|\; o, X_{\mathbf{e}_{k+1}} \notin C \big\}.$$

If $k \in \mathbb{N}_0$, $x \in C \in \widehat{\mathcal{C}}(\mathbf{M}_k, r)$ with $d(\mathbf{M}_k, x) \geq 2r + R_2 + 1$ and $\eta_\infty(x) = 1$, then $\Delta_{k,x} := 2$. Otherwise we set $\Delta_{k,x} := 0$. Then the pseudo-increments are defined as

$$\Delta_k := \max_{x \in V} \Delta_{x,k}.$$

Observe that $\Delta_{k,x} = 2$ implies $\Delta_{l,x} = 0$ for all $l \neq k$, that is, each $x \in V$ causes at most one Δ_k to have value 2. Note also that from time $\mathbf{e}_k$ onward it is impossible to switch a lamp at any $x \in C \in \widehat{\mathcal{C}}(\mathbf{M}_j, r)$ with $d(\mathbf{M}_j, x) \geq 2r + R_2 + 1$ for all $j < k$. In other words, the values Δ_j for $j < k$ depend only on the process up to time $\mathbf{e}_k$.

When computing $|Z_n|$ for $n \geq \mathbf{e}_k$, the pseudo-increment Δ_j, $j < k$, is a lower bound for the distance increase caused by the possibly necessary deviation into some component $C \in \widehat{\mathcal{C}}(\mathbf{M}_j, r)$, which may be needed to reach Z_n with start at $(\mathbf{0}, o)$. As shown above, if $\Delta_j = 2$ then $|Z_n| \geq d(o, X_n) + 2$. As the balls $B(\mathbf{M}_j, r)$, $j < k$, are pairwise disjoint, we obtain for $k \geq 1$, analogously to the previous section,

$$\frac{|Z_{\mathbf{e}_k}|}{\psi(k)} \geq 1 + \frac{1}{\psi(k)} \sum_{j=0}^{k-1} \Delta_j. \tag{6.7}$$

Our next aim is to bound $\mathbb{P}[\Delta_k = 2]$, and thus $\mathbb{E}[\Delta_k]$, uniformly from below by a non-zero constant.

Proposition 6.7. *There is $B_2 > 0$ such that $\mathbb{P}[\Delta_k = 2] \geq B_2$ for all $k \in \mathbb{N}_0$.*

Proof. We sum over all possibilities to hit the annulus

$$\mathcal{R}_k := \{x \in V \mid \psi(k) + r \leq d(o, x) < \psi(k) + r + R_1\},$$

which must be hit almost surely. The hitting point of this set should become $\mathbf{M}_k$ under our construction. Furthermore, denote by $E(n)$, $n \in \mathbb{N}_0$, the event that there are $C, D \in \mathcal{C}(X_n, r)$ with $o \notin C, D$ and $C \neq D$ such that

- from time n onwards the lamplighter walks inside $B(X_n, r) \cup C$ to some $y \in C$ with $d(X_n, y) \geq 2r + R_2 + 1$ without switching the lamp at y during this walk,
- then switches the lamp at y and walks inside $B(X_n, r) \cup C$ back to X_n without switching the lamp at y any more,
- followed by exiting the ball $B(X_n, r)$ into D, from where it does not return to $B(X_n, r)$.

See Figure 6.3. Observe that the assumption $\Lambda \geq 2$ ensures that C and D can be chosen in the required way such that $\mathbb{P}[E(n)] > 0$ for all $n \in \mathbb{N}_0$. We now want to give a uniform lower bound for the probability of $E(n)$. Write $s := 2r + R_2 + 1$. For each $C \in \mathcal{C}(X_n, r)$ there is at least one vertex $y \in C$ with $s \leq d(X_n, y) < s + R_1$, as C is infinite. By Lemma 6.2, there is a path of length at most $\bar{\kappa} := \lfloor (s + R_1)/r_1 \rfloor$ from X_n to y, which lies completely inside $B(X_n, r) \cup C$. Hence, the probability of walking inside $B(X_n, r) \cup C$ from X_n to some y without switching any lamps during this walk is at least $(\varepsilon_0 \, \zeta^2)^{\bar{\kappa}}$. The probability of walking inside $B(X_n, r) \cup C$ from y to X_n without switching any lamps except switching the lamp at X_n in the first step is at least $\varepsilon_0^{\bar{\kappa}} \cdot \xi_0 \cdot \zeta^{2\bar{\kappa}-1}$, as $\mu_y(\eta) \geq \xi_0$ for some $\eta \in \mathcal{N}$ with $\eta(y) = 1$. Finally, the

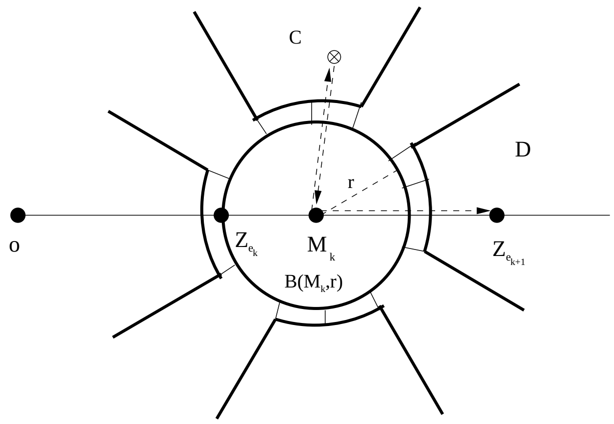


Figure 6.3: Ball around $\mathbf{M}_k$

probability of walking from X_n to the outside of $B(X_n, r)$ with no return to $B(X_n, r)$ is at least $\varepsilon_0^{\bar{r}} \cdot (1 - \bar{F})$, where $\bar{r} := \lfloor (r + R_1)/r_1 \rfloor$. Thus, we obtain

$$\begin{aligned}
\mathbb{P}[E(n)] &\geq \sum_{x \in V} \mathbb{P}[X_n = x] \cdot \left(\varepsilon_0 \,\zeta^2\right)^{\bar{\kappa}} \cdot \varepsilon_0^{\bar{\kappa}} \cdot \xi_0 \cdot \zeta^{2\bar{\kappa}-1} \cdot \varepsilon_0^{\bar{r}} \cdot (1 - \bar{F}) \\
&= \varepsilon_0^{2\bar{\kappa}+\bar{r}} \cdot \xi_0 \cdot \zeta^{4\bar{\kappa}-1} \cdot (1 - \bar{F}) =: B_2. \qquad (6.8)
\end{aligned}$$

In particular, $\mathbb{P}[E(n) \mid X_n = x] \geq B_2$ for all $x \in V$. Denote by T_k the time of the first visit in the annulus $\mathcal{R}_k$. It is $\mathbb{P}[T_k < \infty] = 1$ and $\eta_{T_k}(y) = 0$ for all $y \in C \in \mathcal{C}(X_{T_k}, r)$ with $o \notin C$ and $d(X_{T_k}, y) \geq 2r + R_2 + 1$. Recall in the following that any path inside $B(\mathbf{M}_k, r)$ lies in $\{x \in V \mid d(o, x) \geq \psi(k)\}$ and that $\mathbf{M}_k = X_n$, if $T_k = n$ and the event $E(n)$ holds. Now we can conclude:

$$\begin{aligned}
&\mathbb{P}[\Delta_k = 2] \\
\geq\ &\sum_{x \in \mathcal{R}_k} \sum_{n \geq 0} \sum_{\substack{x_1, \ldots, x_{n-1} \in V, \\ d(o, x_i) < \psi(k) + r}} \mathbb{P}[X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = x, E(n)] \\
\geq\ &\sum_{x \in \mathcal{R}_k} \sum_{n \geq 0} \sum_{\substack{x_1, \ldots, x_{n-1} \in V, \\ d(o, x_i) < \psi(k) + r}} \mathbb{P}[X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = x] \cdot B_2 \\
=\ &\mathbb{P}[T_k < \infty] \cdot B_2 = B_2 > 0.
\end{aligned}$$

□

The last proposition yields:

Corollary 6.8. *We have $\mathbb{E}[\Delta_k] \geq 2 \cdot B_2$ for all $k \in \mathbb{N}_0$.*

□

Now we can prove:

Theorem 6.9. *For the lamplighter random walk on the transitive, connected, locally finite graph* G *with infinitely many ends, assuming that the base random walk on* G *does not converge to a deterministic end and* $\delta_{\mathcal{L}} = 0$, *we have*

$$\ell \geq \Big(1 + \frac{2 \cdot B_2}{2r + R_1 + 1}\Big) \cdot \ell_0 > \ell_0,$$

where B_2 *is given by equation (6.8).*

Proof. Define $A_n := \Delta_n/(2r + R_1 + 1)$. In view of Proposition 6.7, the rest follows analogously to the proof of Theorem 6.6. □

6.2.1.2 Case $\Lambda = 1$

Assume now $\Lambda = 1$, that is, the random walk's projection onto G converges to a deterministic end $\omega \in \partial\mathsf{G}$ almost surely; compare with Woess [42]. In this case it is not ensured that the component D in the proof of Proposition 6.7 can be chosen in the required way such that $\mathbb{P}[E(n)] > 0$. Thus, we have to construct $\mathbf{M}_k$ in a different way.

For each $k \in \mathbb{N}_0$ there is exactly one $D_k \in \mathcal{C}\big(o, \psi(k)\big)$ such that the equation $\mathbb{P}\big[\exists m \forall n \geq m : Z_n \in D_k\big] = 1$ holds. Let

$$E_k := \Big\{x \in D_k \;\Big|\; \psi(k) + r < d(o,x) \leq \psi(k) + r + R_1\Big\}.$$

We replace the midpoint times of the last subsection by the hitting times of the sets E_k:

$$\mathbf{m}_k := \min\big\{n \in \mathbb{N} \;\big|\; X_n \in E_k\big\},$$

which is almost surely finite. Write again $\mathbf{M}_k := X_{\mathbf{m}_k}$. This construction ensures that $B(\mathbf{M}_k, r) \subseteq D_k \cap B\big(o, \psi(k+1)\big)$ and that there is exactly one component $C \in \mathcal{C}(\mathbf{M}_k, r)$ with $D_{k+1} \subseteq C$. We write $\widehat{C}(\mathbf{M}_k, r)$ for the set of all $C \in \mathcal{C}(\mathbf{M}_k, r)$ with $o \notin C$ and $D_{k+1} \nsubseteq C$. The pseudo-increments are now given by

$$\Delta_k := \begin{cases} 2, & \text{if there is } C \in \widehat{C}(\mathbf{M}_k, r) \text{ and } x \in C \\ & \text{with } d(\mathbf{M}_k, x) \geq 2r + R_2 + 1 \text{ and } \eta_\infty(x) = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Each vertex $x \in V$ may induce at most one of Δ_k to have value 2. Moreover, the lamp at any $x \in C \in \widehat{C}(\mathbf{M}_k, r)$ with $d(\mathbf{M}_k, x) \geq 2r + R_2 + 1$ can not be switched, if the lamplighter stands at some vertex in D_{k+1}. Thus, the value of Δ_k is determined if the lamplighter leaves $B(\mathbf{M}_k, r)$ with no further visit to this ball. We get:

Lemma 6.10. *For all $k \in \mathbb{N}_0$,*

$$\mathbb{P}[\Delta_k] \geq B_2 > 0,$$

where B_2 is given by equation (6.8).

Proof. The random walk hits the set E_k almost surely. The rest follows analogously to the proof of Proposition 6.7. □

Using equation (6.5), we can conclude:

Theorem 6.11. *For the lamplighter random walk on a transitive, connected, locally finite graph G with infinitely many ends, assuming that the base random walk converges to a deterministic end and $\delta_{\mathcal{L}} = 0$, we have*

$$\ell \geq \Big(1 + \frac{2 \cdot B_2}{2r + R_1 + 1}\Big) \cdot \ell_0 > \ell_0,$$

where B_2 is given by equation (6.8).

Proof. Define the exit times as in equation (6.4). Then equations (6.5) and the inequality (6.7) hold also in the case $\Lambda = 1$. Analogously to the proof of Theorem 6.6 and 6.9 we obtain the proposed claim. □

6.2.2 Two-ended Graphs

In this section, we want to prove $\ell > \ell_0$ under the assumptions that G has exactly two ends and that all edges of G have weight 1. The latter assumption is needed to create deviations yielding larger distances with positive probability. If we drop this assumption of uniform weight 1 on the edges and also the basic assumption that the subgroup $\Gamma \subseteq \mathrm{AUT}(\mathsf{G}, w)$, which acts weight-transitively on G, preserves transition probabilities, we can construct a counterexample such that the proposed inequality does not hold; see the end of this section. Furthermore, we assume that $\ell_0 > 0$ and that the uniform vertex degree $\deg(\mathsf{G})$ is at least 3. If the uniform vertex degree is 2, then G becomes an infinite line, on which the Switch-Walk-Switch lamplighter random walk has the same speed as its projection onto G; see Example 5.2.

We adjust the considerations of the previous section. There is $r \in \mathbb{N}_0$ such that $\mathsf{G} \setminus B(o, r)$ has exactly two infinite, connected components, denoted by their set of vertices C_1 and C_2. The ends lying in C_1 and C_2 are denoted by ξ_1 and ξ_2, which are represented by one-way infinite paths $\big[o, x_1^{(1)}, x_2^{(1)}, \dots\big]$ and $\big[o, x_1^{(2)}, x_2^{(2)}, \dots\big]$ such that $d\big(o, x_n^{(i)}\big) = n$ for $i \in \{1, 2\}$ and $n \in \mathbb{N}$. For $k \geq 3r + 2$, the reduced graph $\mathsf{G} \setminus B\big(x_{k-1-r}^{(i)}, r\big)$ has

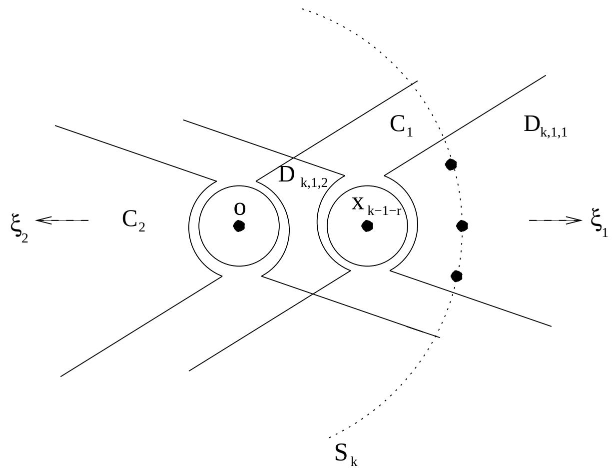


Figure 6.4: Each path from o to ξ_i has to pass through $B\big(x^{(i)}_{k-1-r}, r\big)$.

also exactly two infinite, connected components, denoted by $D_{k,i,1}, D_{k,i,2}$ with ξ_i ending up in $D_{k,i,i}$. Observe that $B\big(x^{(i)}_{k-1-r}, r\big) \cup D_{k,i,i} \subseteq C_i$. Thus, $\operatorname{card} C_i \cap D_{k,i,j} < \infty$ and $o \in D_{k,i,j}$ with $j \in \{1,2\} \setminus \{i\}$. For $k \in \mathbb{N}$, we write $S_k := \{x \in V \mid d(o,x) = k\}$. See Figure 6.4. We now want to show that for $k \geq 3r+2$ the set $S_k \cap D_{k,i,i}$ has at least two elements and its diameter can be uniformly bounded by a constant.

Lemma 6.12. *If* G *is a two-ended, connected, transitive graph* G *with uniform vertex degree* $\deg(\mathsf{G}) \geq 3$*, then* $r > 0$.

Proof. Assume $r = 0$. We want to show that this implies that G has more than two ends. If $r = 0$, then $\mathsf{G} \setminus \{x\}$ has two infinite, connected components for each $x \in V$, identified by their set of vertices D_1, D_2; see Figure 6.5. As $\deg(\mathsf{G}) \geq 3$, there are three neighbours $y_1, y_2, y_3 \in V$ of x such that w.l.o.g. $y_1 \in D_1$ and $y_2, y_3 \in D_2$. Furthermore, there is a path inside D_2 from y_2 to y_3. The reduced graph $\mathsf{G} \setminus \{y_2\}$ has also two infinite, connected components, denoted by D_3, D_4. Observe that w.l.o.g. $D_1 \cup \{x\} \subseteq D_3$. Analogously, $\mathsf{G} \setminus \{y_3\}$ has two infinite, connected components D_5, D_6 such that w.l.o.g. $D_1 \cup D_4 \cup \{x\} \subseteq D_5$. Observe that D_1, D_4 and D_6 are pairwise disjoint. Thus, $\mathsf{G} \setminus B(x,1)$ has at least three infinite connected components. This yields the proposed claim, as G is assumed to have exactly two ends. ∎

The last lemma yields that $\operatorname{card} S_k \cap D_{k,i,i} \geq 2$ for $k \geq 3r+2$: if it is $S_k \cap D_{k,i,i} = \{x\}$ for some $x \in V$, then each path $\omega \in \xi_i$ has to pass through x. But this means that $\mathsf{G} \setminus \{x\}$ would have two infinite, connected components, providing $r = 0$. This is a contradiction to the previous lemma.

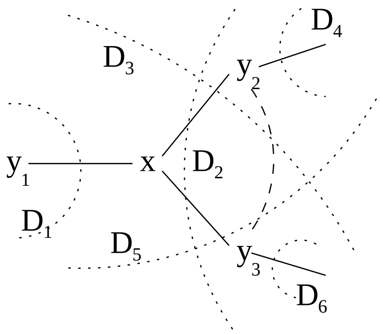


Figure 6.5: Sketch to the proof of Lemma 6.12

Lemma 6.13. *If $k \geq 3r+2$ and $x \in S_k \cap D_{k,i,i}$, then $d\big(x, x^{(i)}_{k-1-r}\big) \leq 3r+1$.*

Proof. For $k \geq 3r+2$, consider the ball $B_{k,i} := B\big(x^{(i)}_{k-1-r}, r\big)$. By construction of $B_{k,i}$ and $D_{k,i,i}$, each path from o to any $x \in S_k \cap D_{k,i,i}$ has to pass through the ball $B_{k,i}$. Consider now a shortest path $\omega = [\omega_0, \ldots, \omega_k]$ from o to some vertex $\omega_k \in S_k \cap D_{k,i,i}$. Let ω_t be the last vertex of ω lying inside $B_{k,i}$. The vertex ω_{t+1}, the first vertex of ω lying outside $B_{k,i}$ after the visit in $B_{k,i}$, has a distance to o of at least $k-2r$. Thus,

$$d\big(x, x^{(i)}_{k-1-r}\big) \leq d(x, \omega_{t+1}) + d(\omega_{t+1}, \omega_t) + d\big(\omega_t, x^{(i)}_{k-1-r}\big) \leq 2r+1+r = 3r+1.$$

□

From the proof of the last lemma it follows that the diameter of each $S_k \cap D_{k,i,i}$ can be uniformly bounded by $6r+2$. Now the exit times are defined as

$$\mathbf{e}_k := \min\big\{m \in \mathbb{N} \;\big|\; \forall n \geq m : d(o, X_n) \geq k + R_2 + 1\big\}$$

for $k \geq 3r+2$. Furthermore,

$$\Delta_k := \begin{cases} 1, & \text{if } X_\infty = \xi_i \text{ implies} \\ & \exists x, y \in S_k \cap D_{k,i,i}, x \neq y, \text{ with } \eta_\infty(x) = \eta_\infty(y) = 1 \\ 0, & \text{otherwise} \end{cases}.$$

When computing $|Z_n|$ with $n \geq \mathbf{e}_k$, Δ_k is again a lower bound for the distance increase caused by a possibly necessary deviation from one point in $S_k \cap D_{k,i,i}$ to another one in the same set, because $\Delta_k = 1$ implies that a shortest tour from $(\mathbf{0}, o)$ to Z_n causes the lamplighter to visit $S_k \cap D_{k,i,i}$ at least twice. Observe that the value of Δ_k depends only on the process up to time $\mathbf{e}_k$. Analogously to the previous section, we get for $k \geq 3r+2$

$$\frac{|Z_{\mathbf{e}_k}|}{k+R_2+1} \geq 1 + \frac{1}{k+R_2+1} \sum_{j=3r+2}^{k} \Delta_j.$$

We now give a lower bound for $\mathbb{P}[\Delta_k = 1]$:

Lemma 6.14. *There is $B_3 > 0$ such that for $k \geq 3r+2$*

$$\mathbb{P}[\Delta_k = 1] \geq B_3.$$

Proof. Initial remark before the proof: it can be shown that $(X_n)_{n\in\mathbb{N}_0}$ converges almost surely to a deterministic end. However, we shall not need this fact in the proof of this lemma.

First,

$$\begin{aligned} \mathbb{P}[\Delta_k = 1] &= \mathbb{P}[X_\infty = \xi_1] \cdot \mathbb{P}\big[\Delta_k = 1 \mid X_\infty = \xi_1\big] \\ &\quad + \mathbb{P}[X_\infty = \xi_2] \cdot \mathbb{P}\big[\Delta_k = 1 \mid X_\infty = \xi_2\big]. \end{aligned} \tag{6.9}$$

Assume now w.l.o.g. that $[X_\infty = \xi_1]$ has positive probability and we assume that this event holds. We construct a tour on this event to establish $\Delta_k = 1$ with positive probability independently of k. The set $S_k \cap D_{k,1,1}$ is hit almost surely on the event $[X_\infty = \xi_1]$ and we denote by T_k its hitting time. Assume now that x is the hitting point. Then there is a path of length at most $6r+2$ to some other point $y \in S_k \cap D_{k,1,1}$. Thus, walking from x with configuration η to y with – if necessary – switching the lamps at x and y on and without switching any further lamp on the route to y has a probability of at least

$$\left(\frac{\xi_0}{\zeta}\right)^{\eta(x)} \cdot (\varepsilon_0\,\zeta^2)^{6r+2} \cdot \left(\frac{\xi_0}{\zeta}\right)^{\eta(y)}.$$

From y it is possible to reach the vertex $w := x^{(1)}_{k+R_2+1} \in S_{k+R_2+1} \cap C_1$ on a path of length at most $4r+3+R_2$, as

$$d(y,w) \leq d\big(y, x^{(1)}_{k-r-1}\big) + d\big(x^{(1)}_{k-r-1}, w\big) \leq (3r+1) + (r+2+R_2).$$

Thus, the probability of walking from y to w without switching any lamps is at least $(\varepsilon_0\,\zeta^2)^{4r+3+R_2}$. The ball $B(w,r)$ can be left into the infinite component not containing o on the path $[w, x^{(1)}_{k+R_2+2}, \ldots, x^{(1)}_{k+R_2+r+2}]$ of length $r+1$. Having left this ball into this component, the lamplighter returns into $B(w,r)$ with a probability strictly smaller than 1, namely with a probability of at most $\widehat{F}$, where

$$\widehat{F} := \max\Big\{F\big(v, B(o,r)\big) \mid v \in S_{r+1}, F\big(v, B(o,r)\big) < 1\Big\} < 1.$$

Observe that the inclusion of events $[X_\infty = \xi_1] \subseteq [T_k < \infty]$ holds, providing

$$\sum_{x \in S_k \cap D_{k,1,1}} \sum_{n \in \mathbb{N}} \mathbb{P}\big[T_k = n, X_n = x\big] = \mathbb{P}[T_k < \infty] \geq \mathbb{P}[X_\infty = \xi_1].$$

Thus, we can estimate:

$$\begin{aligned}
& \mathbb{P}\big[\Delta_k = 1 \mid X_\infty = \xi_1\big] \\
\geq\ & \frac{1}{\mathbb{P}[X_\infty = \xi_1]} \sum_{x \in S_k \cap D_{k,1,1}} \sum_{n \in \mathbb{N}} \mathbb{P}\big[T_k = n, X_n = x\big] \\
& \cdot \Big(\frac{\xi_0}{\zeta}\Big)^{\eta(x)} \cdot (\varepsilon_0 \zeta^2)^{6r+2} \cdot \Big(\frac{\xi_0}{\zeta}\Big)^{\eta(y)} \cdot (\varepsilon_0 \zeta^2)^{4r+3+R_2} \cdot \varepsilon_0^{r+1} \cdot \big(1 - \widehat{F}\big) \\
\geq\ & \xi_0^2 \cdot \varepsilon_0^{11r+6+R_2} \cdot \zeta^{20r+8+2R_2} \cdot \big(1 - \widehat{F}\big) =: B_3. \qquad (6.10)
\end{aligned}$$

This together with equation (6.9) finishes the proof. ☐

The last lemma yields that $\mathbb{E}[\Delta_k] \geq B_3$. Now we can state:

Theorem 6.15. *For the Switch-Walk-Switch lamplighter random walk on a transitive, connected, locally finite graph* G *with uniform vertex degree of at least* 3*, exactly two ends and uniform weight* 1 *on its edges, assuming* $\ell_0 > 0$ *and* $\delta_{\mathcal{L}} = 0$*,*

$$\ell \geq \big(1 + B_3\big) \cdot \ell_0 > \ell_0,$$

where B_3 *is given by equation (6.10).*

Proof. The proof works analogously to the proofs of Theorems 6.6 and 6.9. ☐

Besides the two-way-infinite line we give another counterexample where the Switch-Walk-Switch lamplighter random walk has same speed as its projection onto the base graph:

Example 6.16: Consider the 'ladder' given by the Cayley graph of $\mathbb{Z} \times \mathbb{Z}_2$ w.r.t. the generators $(\pm 1, 0)$ and $(0, 1)$. For $z \in \mathbb{Z}$ assign the weights as follows:

$$\begin{aligned}
w\big((2z, 0), (2z+1, 0)\big) &= 1, \\
w\big((2z+1, 0), (2z+2, 0)\big) &= 5, \\
w\big((2z, 1), (2z+1, 1)\big) &= 5, \\
w\big((2z+1, 1), (2z+2, 1)\big) &= 1, \\
w\big((z, 0), (z, 1)\big) &= 1.
\end{aligned}$$

Then this weighted graph is transitive. The associated metric is denoted by $d_{\mathbb{Z} \times \mathbb{Z}_2}(\cdot, \cdot)$. A minimal weighted path from $(0, 0)$ to $(z, m) \in \mathbb{Z} \times \mathbb{Z}_2$ visits

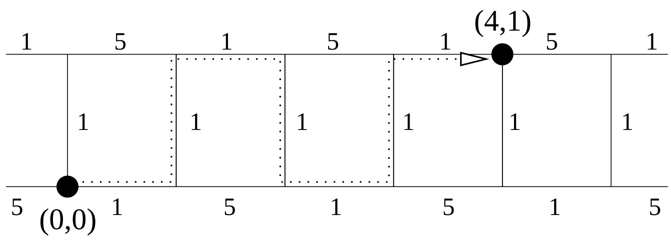


Figure 6.6: A minimal-weighted path from $(0,0)$ to $(4,1)$ in $\mathbb{Z} \times \mathbb{Z}_2$

all points (z', m') with $0 < |z'| < |z|$, $\operatorname{sgn}(z) = \operatorname{sgn}(z')$ and $m' \in \mathbb{Z}_2$. See Figure 6.6.

Let $p \in (1/4; 1/2)$. We equip $\mathbb{Z} \times \mathbb{Z}_2$ with a transient random walk defined by the following transition probabilities:

$$\begin{aligned} p_0\big((z,z'),(z+1,z')\big) &:= p, \quad p_0\big((z,z'),(z-1,z')\big) := \frac{1}{2} - p, \\ p_0\big((z,z'),(z,z'+1)\big) &:= \frac{1}{2} \quad \text{for } z \in \mathbb{Z}, z' \in \mathbb{Z}_2. \end{aligned}$$

Observe that there is no subgroup Γ' of $\mathrm{AUT}(\mathbb{Z} \times \mathbb{Z}_2, w)$ acting transitively on $\mathbb{Z} \times \mathbb{Z}_2$ such that the transition probabilities are Γ'-invariant. Recall that we assumed this property throughout our computations. Define now $\|(z,z')\| := |z| + z'$ with z' interpreted as an element of $\mathbb{Z}$. The base random walk starting in $(0,0)$ is again denoted by $(X_n)_{n \in \mathbb{N}_0}$ and the associated lamplighter random walk by $(\eta_n, X_n)_{n \in \mathbb{N}_0}$. As an easy consequence of *Kingman's subadditive ergodic theorem*, $\|X_n\|/n$ converges almost surely to some positive constant $c \in (0;1)$. Observe now that

$$2\,\|X_n\| - 1 \leq d_{\mathbb{Z} \times \mathbb{Z}_2}\big((0,0), X_n\big) \leq 2\,\|X_n\| + 1.$$

Thus, $\lim_{n \to \infty} d_{\mathbb{Z} \times \mathbb{Z}_2}\big((0,0), X_n\big)/n = 2c$ almost surely. One can show analogously to Example 5.2 that $|(\eta_n, X_n)|/n$ converges also to $2c$ almost surely, that is, the lamplighter does not escape faster than its projection onto $\mathbb{Z} \times \mathbb{Z}_2$.

6.3 The Case $\ell_0 = 0$

We explain in this section that also lamplighter random walks arising from transient base random walks with zero speed escape with non-zero speed to infinity. For transient base random walks on finitely generated groups with zero drift w.r.t. the word metric, Dyubina [9] proved non-zero drift of the lamplighter random walk. Here, we explain the acceleration of the lamplighter for our more general case of graphs and (pseudo-) metrics.

Denote by $\pi_n(\eta, x) := p^{(n)}\big((\mathbf{0}, o), (\eta, x)\big)$ the n-step transition probability measure of the lamplighter random walk. Recall the following definitions:

(i) The entropy of the probability measure π_n is

$$H(\pi_n) := -\sum_{(\eta,x)\in\mathcal{N}\times V} \pi_n(\eta, x)\, \log \pi_n(\eta, x).$$

(ii) The asymptotic entropy is

$$h := \lim_{n\to\infty} \frac{H(\pi_n)}{n}.$$

(iii) The growth rate of $\mathbb{Z}_2 \wr \mathsf{G}$ is

$$\mathrm{gr}\big(\mathbb{Z}_2 \wr \mathsf{G}\big) := \lim_{n\to\infty} \frac{\log \mathrm{card}\, B(o, n)}{n}.$$

In general, the entropy and the growth rate exist for random walks on transitive graphs; see Kaimanovich and Woess [17, Lemma 5.2] together with Kingman's subadditive Ergodic Theorem [20]. We need the following essential result about entropy applied to our case:

Proposition 6.17. *It is*

$$h \le \ell \cdot \mathrm{gr}\big(\mathbb{Z}_2 \wr \mathsf{G}\big).$$

Moreover, $h = 0$ if and only if its Poisson boundary is trivial.

Proof. See Kaimanovich and Woess [17, Theorems 4.7, 5.3], where this is proved in a straightforward way only for the case that $d(\cdot, \cdot)$ is the natural graph metric. The proof adapts to our metric with weights. □

Now we can conclude:

Theorem 6.18. *For a transitive, connected, locally finite graph G equipped with a transient, space-homogeneous random walk, the associated lamplighter random walk has non-zero drift.*

Proof. The mapping

$$(\eta, x) \mapsto \mathbb{P}_{(\eta,x)}[\eta_\infty(o) = 0]$$

defines a non-constant bounded harmonic function. Thus, the Poisson boundary is non-trivial, that is, $h > 0$; see Kaimanovich [14, Section 1.3.1] and Kaimanovich and Woess [17, Theorem 4.7]. Proposition 6.17 finishes the proof. □

Observe that we do not need the assumption from the previous section that G must have at least two ends.

6.4 Remarks

6.4.1 Switch-Walk-Switch Random Walk

Consider a transitive, connected, locally finite graph G equipped with a nearest neighbour random walk P_0 in the sense of Section 5.1 such that $\ell_0 > 0$. We assign to each edge the weight 1 yielding that the corresponding distance measure is the natural graph metric. Choose $p \in (0,1)$. Let $\mu_x(\mathbb{1}_x) = p$ and $\mu_x(\mathbf{0}) = 1-p$ for every $x \in V$, yielding $R_2 = 0$. Then the corresponding lamplighter random walk is as follows: the lamplighter tosses a coin for deciding whether to switch the lamp at his actual position with probability p, followed by a walking step to a random neighbour vertex, followed by tossing a coin again whether to switch the lamp at his destination vertex. For $(\eta, x) \in \mathcal{N} \times V$, the distance $|(\eta,x)|$ is then given by the length of a shortest path from $(\mathbf{0}, o)$ to (η, x) inside $\mathbb{Z}_2 \wr \mathsf{G}$, if we set $\delta_{\mathcal{L}} = 1$. In the case $\delta_{\mathcal{L}} = 0$ the distance of $(\mathbf{0}, o)$ to (η, x) is given by $|(\eta,x)| - \mathrm{supp}(\eta)$.

In the case $\delta_{\mathcal{L}} = 1$ we get

$$\ell \geq \Big(1 + \min\big\{p/(1-p), 1\big\}\, \varepsilon_0\, (1-p)\, \tilde{p}\Big) \cdot \ell_0,$$

where $\tilde{p} = \mathbb{P}_{(\mathbf{0},y)}[\forall n \geq 1 : X_n \neq o]$ for some $y \in V$ such that $y \sim o$ and $\tilde{p} > 0$. In the case $\delta_{\mathcal{L}} = 0$ the set $\big\{x \in V \mid d(o,x) = \psi(k) + r\big\}$ is hit almost surely. Thus, assumming additionally that G has infinitely many ends, we obtain analogously

$$\ell \geq \Big(1 + \frac{4}{2r+1}\, \varepsilon_0^{5r+3}\, p\, (1-p)\, (1 - \bar{F})\Big) \cdot \ell_0. \tag{6.11}$$

6.4.2 Walk-or-Switch Random Walk

There is another typical way to define a lamplighter random walk by the following transition probabilities, where $p \in (0,1)$:

$$p\big((\eta,x),(\eta',x')\big) := \begin{cases} p \cdot \mu_x(\eta \oplus \eta'), & \text{if } x = x' \\ (1-p) \cdot p_0(x,x'), & \text{otherwise} \end{cases}.$$

At each step the lamplighter tosses a coin and decides either to switch some lamps in the neighbourhood of his actual position *or* to walk to a random neighbour vertex. The Theorems 6.6, 6.9, 6.11, 6.15 and 6.18 hold analogously except from the adjustion of the constants b_1, B_2 and B_3. In Chapter 7, we will use this model to investigate a lamplighter random walk on the homogeneous tree.

6.4.3 Multi-State Lamps

The presented techniques for proving the acceleration of the lamplighter random walks can also be applied in the case that there are more possible lamp states encoded by elements of $\mathbb{Z}/r\mathbb{Z}$ with $r > 2$. In this case one may assign weights $w_j \geq 0$ to edges of pairs of neighbours of the form $(\eta, x) \sim (\eta \oplus (j\mathbb{1}_x), x)$ for $j \in \mathbb{Z}/r\mathbb{Z}$, $j \neq 0$. Then the Theorems 6.6, 6.9, 6.11, 6.15 and 6.18 hold analogously except for the necessity to adjust the constants b_1, B_2 and B_3 and to replace $\delta_{\mathcal{L}}$ by $\min w_j$.

6.4.4 Markovian Distance

For $(\eta, x) \in \mathcal{N} \times V$, the Markovian distance $|(\eta, x)|_{\mathbb{P}}$ is given by

$$\min \left\{ \sum_{i=1}^{m} w(x_{i-1}, x_i) \;\middle|\; \begin{array}{c} [o, x_1, \ldots, x_{m-1}, x] \text{ is a path from } o \text{ to } x \text{ such that} \\ \mathbb{P}_{(\mathbf{0},o)}\big[X_1 = x_1, \ldots, X_m = x_m, \eta_m = \eta\big] > 0 \end{array} \right\}.$$

Then the limit $\ell_{\mathbb{P}} := \lim_{n\to\infty} |Z_n|_{\mathbb{P}}/n$ exists also almost surely and is almost surely constant. The proofs of Section 6.2 adapt also to this case and we obtain the inequality $\ell_{\mathbb{P}} > \ell_0$, if G has infinitely many ends. Theorem 6.18 holds also, if one considers the drift w.r.t. the Markovian distance. However, if G has two ends, then it is possible to get $\ell = \ell_0$ in the case $\delta_{\mathcal{L}} = 0$: e.g., consider the Cayley graph of $\mathbb{Z} \times \mathbb{Z}_2$ w.r.t. the set of generators $(\pm 1, 0), (0, 1)$, uniform weight 1 on the edges and $\mu_z(\mathbb{1}_y) = 1/4$, where $y = z$ or $y \sim z$ for all $z \in \mathbb{Z} \times \mathbb{Z}_2$.

6.4.5 Greenian Distance

Another metric on G is given by the Greenian distance

$$d_{\mathrm{Green}}(x, y) := -\ln \mathbb{P}_x[T_y < \infty],$$

where T_y is the hitting time of $y \in V$. We can define the Greenian metric on $\mathbb{Z}_2 \wr \mathsf{G}$ analogously. These metrics are no path metrics induced by weights on the edges. Blachère, Haïssinsky and Mathieu [3] proved that the entropy and the rate of escape w.r.t. the Greenian distance of random walks on groups are equal. If G is the Cayley graph of a group Γ and the random walk on G is governed by a probability measure μ with $\operatorname{supp} \mu = \Gamma$, then the entropy of a lamplighter random walk on $\mathbb{Z}_2 \wr \Gamma$ is strictly bigger than the entropy of the random walk's projection onto G, because the Poisson boundary of the lamplighter random walk projects non-trivially onto the one of the random walk on the base graph; compare with Kaimanovich and Vershik [15, Theorem 3.2]. It follows that also w.r.t. the Greenian distance the lamplighter random walk is faster than its projection onto G.

6.5 Summary

We considered a transitive, connected, locally finite graph G, where a pseudometric on G arises from the weights on the edges. We proved that – under suitable assumptions on G – a lamplighter random walk on G has bigger drift than the random walk's projection onto G. In general, this is not true. Assuming non-zero drift of the base random walk, we proved in Section 6.1 that the lamplighter escapes always faster than its projection onto G, if edges in $\mathbb{Z}_2 \wr \mathsf{G}$ corresponding to lamp switches have positive weight, that is, if lamp switches are charged by the metric. In Section 6.2, we proved the lamplighter's acceleration for the case that those edges have weight zero: for graphs with infinitely many ends, this was shown in Section 6.2.1, and for two-ended graphs with the additional assumption of uniform weight 1 on all edges, the acceleration was proved in Section 6.2.2. The acceleration of the lamplighter random walk arising from a base random walk with zero drift was explained in Section 6.3, where we made no restrictions to the weights and the number of ends of G. The proofs of this chapter adapt also partially to other metrics and lamplighter random walk models, as explained in Section 6.4.

Chapter 7

Lamplighter Tree

In this chapter we investigate lamplighter random walks arising from a simple random walk on homogeneous trees. We will construct lower and upper bounds for the rate of escape of the 'Walk-or-Switch' (WoS) lamplighter random walk and a lower bound for the 'Switch-Walk-Switch' (SWS) lamplighter random walk. The plan for this chapter is as follows: in Section 7.1, we explain the special algebraic structure of the lamplighter tree and its WoS random walk. In Section 7.2, we construct a lower and an upper bound for the WoS lamplighter random walk's drift, while in Section 7.3 we construct another lower bound. Finally, in Section 7.4 we give a thighter lower bound for the associated SWS lamplighter random walk than the one given by Theorem 6.9.

7.1 Simple Random Walk on the Lamplighter Tree

Let $3 \leq q \in \mathbb{N}$. Consider the homogeneous tree $\mathcal{T}_q = (V, E)$ of degree q, that is, each vertex has q neighbours. We omit the use of weights on the edges. Let $\mathcal{S} := \{a_1, \ldots, a_q\}$. Then all vertices of $\mathcal{T}_q$ can be described uniquely by finite words over the alphabet $\mathcal{S}$, where no two consecutive letters are equal, such that we obtain the following symmetric neighbourhood property: each $a \in \mathcal{S}$ is adjacent to the empty word o, which is assigned to any vertex; if $w \in V$ with last letter a_i, then wa_j is adjacent to w for every $a_j \in \mathcal{S} \setminus \{a_i\}$. In this case w is closer to o than each wa_j. We can define a *group operation* on V by *concatenation* of words with possible *cancellations* in the middle: if $u, v \in V$ are represented as words over $\mathcal{S}$, then $u \circ v$ is the concatenation with iterated deletions of all blocks of the form '$a_i a_i$'. For instance, if $u = a_1 a_2 a_1$, $v = a_1 a_2 a_3$, then $u \circ v = a_1 a_3$. In particular, the identity is o and we have $a_i^{-1} = a_i$ for all $i \in \{1, \ldots, q\}$. With this definition $\mathcal{T}_q$ is the Cayley graph of the *free product group* $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ of q factors $\mathbb{Z}_2$; compare with Section 3.3.

In the sequel, we shall identify $\mathcal{T}_q$ with this group. The distance $d(x, y)$ of two points $x, y \in \mathcal{T}_q$ is the distance w.r.t. the natural graph metric on the tree.

Furthermore, assume again that there sits a lamp at each vertex of $\mathcal{T}_q$, which can be switched off or on, again encoded by '0' and '1'. The set of all finitely supported functions $\eta : \mathcal{T}_q \to \mathbb{Z}_2$ is again denoted by $\mathcal{N}$. The lamplighter graph $\mathbb{Z}_2 \wr \mathcal{T}_q$ is constructed as in Section 5.1 without assigning weights to the edges and we take over the same notation unless it is remarked otherwise. We construct a group such that its Cayley graph is exactly $\mathbb{Z}_2 \wr \mathcal{T}_q$. For this purpose, we now want to define a group operation on the vertex set $\mathcal{N} \times \mathcal{T}_q$. For $x, y \in \mathcal{T}_q$ and $\eta \in \mathcal{N}$, define

$$(x\eta)(y) := \eta(x^{-1}y).$$

The *group operation* on $\mathcal{N} \times \mathcal{T}_q$ is then given by

$$(\eta_1, x)(\eta_2, y) := \big(\eta_1 \oplus (x\eta_2), xy\big),$$

where $x, y \in \mathcal{T}_q$, $\eta_1, \eta_2 \in \mathcal{N}$ and $(\mathbf{0}, o)$ is the identity. With this operation we obtain the wreath product

$$\mathcal{L}_q := \Big(\sum_{x \in \mathcal{T}_q} \mathbb{Z}_2\Big) \rtimes \mathcal{T}_q.$$

Let

$$\mathcal{S}_{\mathcal{L}_q} := \big\{(\mathbb{1}_o, o), (\mathbf{0}, a_i) \;\big|\; a_i \in \mathcal{S}\big\}.$$

The Cayley graph of $\mathcal{L}_q$ with respect to $\mathcal{S}_{\mathcal{L}_q}$ is exactly the graph $\mathbb{Z}_2 \wr \mathcal{T}_q$. Thus, we identify in the sequel $\mathbb{Z}_2 \wr \mathcal{T}_q$ with $\mathcal{L}_q$ and call it the Lamplighter Tree.

The length of a shortest path in the Cayley graph from $(\mathbf{0}, o)$ to (η, x) is again denoted by $|(\eta, x)|$. As we will equip $\mathbb{Z}_2 \wr \mathcal{T}_q$ with a nearest neighbour random walk, $|(\eta, x)|$ is the minimal amount of time needed for the lamplighter to start at o with all lamps off and walk to x with restoring the configuration η. This definition of distances corresponds to the model of Section 6.1 where each edge has weight 1.

We now construct a nearest neighbour lamplighter random walk on $\mathcal{L}_q$ according to Section 6.4.2. Let $p \in (0, 1)$. Consider the sequence of i.i.d. random variables $(\mathbf{i}_k)_{k \in \mathbb{N}}$ valued in $\mathcal{L}_q$, the increments, with distribution

$$\mu(w) = \begin{cases} p, & \text{if } w = (\mathbb{1}_o, o) \\ (1-p)/q, & \text{if } w = (\mathbf{0}, a_i) \text{ for some } a_i \in \mathcal{S} \\ 0, & \text{otherwise} \end{cases}.$$

A lamplighter random walk on $\mathcal{L}_q$ starting at $(\mathbf{0}, o)$ is described by $(Z_n)_{n\in\mathbb{N}_0}$ in the following natural way:

$$Z_0 := (\mathbf{0}, o), \quad Z_n := Z_{n-1}\mathbf{i}_n \text{ for all } n \geq 1. \tag{7.1}$$

The distribution of Z_n is $\mu^{(n)}$, the n-th convolution power of μ with respect to the group structure of $\mathcal{L}_q$. We write again $Z_n = (\eta_n, X_n)$ and denote by $\mathbb{P}_{(\eta,x)}[\,\cdot\,]$ the probability measure that governs the lamplighter random walk starting at $(\eta, x) \in \mathcal{N} \times \mathcal{T}_q$ instead of $(\mathbf{0}, o)$. In the case $(\eta, x) = (\mathbf{0}, o)$ we omit the subindex. Our aim is to give bounds for the lamplighter's rate of escape

$$\ell_{\mathrm{WoS}} = \lim_{n\to\infty} \frac{|Z_n|}{n}.$$

It is well-known that simple random walk on $\mathcal{T}_q$ has rate of escape $(q-2)/q$. Furthermore, we obtain for our random walk:

Lemma 7.1.

$$\ell_0 := \lim_{n\to\infty} \frac{d(o, X_n)}{n} = (1-p)\frac{q-2}{q} \quad \mathbb{P}-a.s.$$

Proof. Standing at some $x \in \mathcal{T}_q \setminus \{o\}$, we move away from o with probability $(1-p)(q-1)/q$ and towards o with probability $(1-p)/q$. Thus, $d(o, X_n)$ is a classical birth-and-death Markov chain on the non-negative integers. Therefore

$$\ell_0 = \lim_{n\to\infty} \frac{d(o, X_n)}{n} = (1-p)\frac{q-1}{q} - \frac{1-p}{q} = (1-p)\frac{q-2}{q} \quad \mathbb{P}-a.s..$$

□

As a consequence, our lamplighter random walk is transient, since the projection $(X_n)_{n\in\mathbb{N}_0}$ onto the tree is transient with non-zero drift.

We now state a lemma which we will use several times in later computations.

Lemma 7.2. *For $y \in \mathcal{T}_q$, let $T_y := \min\{m \geq 1 \mid X_m = y\}$ be the first return stopping time of y. Then:*

1. *If $z = (\eta_x, x) \in \mathcal{L}_q$ and $y \in \mathcal{T}_q$ is a neighbour of x in the tree, then*

$$F := \mathbb{P}_z[T_y < \infty] = \frac{1}{q-1}.$$

2. $$G := \sum_{n\geq 0} \mathbb{P}[X_n = o] = \frac{q-1}{(1-p)(q-2)}.$$

Proof. By vertex-transitivity, it is obvious that $\mathbb{P}_z[T_y < \infty]$ depends only on the neighbourhood property and not on the specific points x and y. So we get the recursive equation

$$F = \mu\big((\mathbf{0}, a_i)\big) + \mu\big((\mathbb{1}_o, o)\big) \cdot F + \sum_{a_j \in \mathcal{S}\setminus\{a_i\}} \mu\big((\mathbf{0}, a_j)\big) \cdot F^2$$

for any $a_i \in \mathcal{S}$, or equivalently,

$$(1-p)\frac{q-1}{q} \cdot F^2 - (1-p) \cdot F + \frac{1-p}{q} = 0.$$

As $(X_n)_{n\in\mathbb{N}_0}$ is transient, $F < 1$ has to be fulfilled. Thus, the right solution of this quadratic equation is $F = 1/(q-1)$.

As

$$\mathbb{P}[T_o < \infty] = \mu\big((\mathbb{1}_o, o)\big) + \sum_{a_i \in \mathcal{S}} \mu\big((\mathbf{0}, a_i)\big) \cdot F = p + \frac{1-p}{q-1},$$

it follows that

$$G = \sum_{n\geq 0} \mathbb{P}[T_o < \infty]^n = \frac{1}{1 - \mathbb{P}[T_o < \infty]} = \frac{q-1}{(1-p)(q-2)}.$$

□

As $(X_n)_{n\in\mathbb{N}_0}$ leaves every finite set almost surely, this sequence converges to a random variable X_∞ valued in $\partial\mathcal{T}_q$, which can be identified with the set of infinite words over the alphabet $\mathcal{S}$ such that no two consecutive letters are equal.

Lemma 7.3. *Let $a \in \mathcal{S}$. Then*

$$\mathbb{P}\big[X_\infty \text{ has first letter } a\big] = \frac{1}{q}.$$

Proof. By conditioning to the last visit to o before finally walking to a with no consecutive visit to o, we obtain

$$\mathbb{P}\big[X_\infty \text{ has first letter } a\big] = G \cdot \mu\big((\mathbf{0}, a)\big) \cdot \big(1 - F\big) = \frac{1}{q}.$$

□

Again, $(\eta_n)_{n\in\mathbb{N}_0}$ converges almost surely pointwise to a random configuration $\eta_\infty : \mathcal{T}_q \to \mathbb{Z}_2$. The cone rooted at $w \in \mathcal{T}_q$ is

$$C_w := \big\{w' \in \mathcal{T}_q \;\big|\; w \text{ is prefix of } w'\big\}.$$

The complement $\mathcal{T}_q \setminus C_w$ is denoted by $\overline{C_w}$. Later computations require the following probabilities:

$$\begin{aligned} \nu_1 &:= \mathbb{P}\big[a_1 \text{ is not first letter of } X_\infty, \eta_\infty(C_{a_1}) \not\equiv \mathbf{0}\big] \quad \text{and} \\ \nu_2 &:= \mathbb{P}\big[a_1 \text{ is first letter of } X_\infty, \eta_\infty(\overline{C_{a_1}}) \equiv \mathbf{0}\big]. \end{aligned}$$

There is a simple relation between ν_1 and ν_2: by vertex-transitivity and Lemma 7.3, we have

$$\nu_1 = F \cdot \mathbb{P}[a_1 \text{ is first letter of } X_\infty, \eta_\infty(\overline{C_{a_1}}) \not\equiv \mathbf{0}] = \frac{1}{q-1} \cdot \Big(\frac{1}{q} - \nu_2\Big).$$

In the next section we will derive a formula for ℓ_{WoS} that depends on ν_1, ν_2 respectively. We will also give lower bounds for these two probabilities, providing upper and lower bounds for ℓ_{WoS}.

7.2 Lower and Upper Bound

In this section we construct a lower and an upper bound for ℓ_{WoS}. In particular, the lower bound will be strictly bigger than the trivial lower bound ℓ_0 given by Lemma 7.1.

We reformulate our problem for finding a formula for ℓ_{WoS}. We proceed analogously to Section 3.3, that is, we prove convergence of the sequence

$$\Big(\mathbb{E}[|Z_{n+1}|] - \mathbb{E}[|Z_n|]\Big)_{n\in\mathbb{N}}$$

and compute its limit, which then must equal ℓ_{WoS}. Rewriting the elements of this sequence we obtain

$$\mathbb{E}[|Z_{n+1}|] - \mathbb{E}[|Z_n|] = \sum_{g\in\mathcal{S}_{\mathcal{L}_q}} \mu(g) \int_{\mathcal{L}_q} \big(|gZ_n| - |Z_n|\big)\, d\mathbb{P}.$$

Define the random variables

$$Y_{g,n} := |gZ_n| - |Z_n|$$

for any given $g \in \mathcal{S}_{\mathcal{L}_q}$ and $n \in \mathbb{N}$. To understand the behaviour of $Y_{g,n}$ for $n \to \infty$, we now investigate differences of the form $|g(\eta,x)| - |(\eta,x)|$. For this purpose, define for $a \in \mathcal{S}$ and $\eta \in \mathcal{N}$ the configurations

$$\eta_a(w) := \begin{cases} \eta(w), & \text{if } w \in C_a \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{\eta_a}(w) := \begin{cases} \eta(w), & \text{if } w \in \overline{C_a} \\ 0, & \text{otherwise} \end{cases}.$$

With this notation we have $\eta = \eta_a \oplus \overline{\eta_a}$.

Proposition 7.4. *Let $a \in \mathcal{S}$, $x \in C_a$ and $\eta \in \mathcal{N}$. Then*

$$\big|(\mathbf{0}, a)(\eta, x)\big| - \big|(\eta, x)\big| = \begin{cases} 1, & \text{if } \overline{\eta_a} \not\equiv \mathbf{0} \\ -1, & \text{if } \overline{\eta_a} \equiv \mathbf{0} \end{cases}.$$

Proof. Write $x = ay$ with $y \in \overline{C_a}$. Since $\eta_a(w) = 1$ if and only if $(a\eta_a)(aw) = 1$ for $w \in \mathcal{T}_q$, we obtain

$$\big|(\eta, x)\big| = \big|(\overline{\eta_a}, o)\big| + \big|(\eta_a, ay)\big| = \big|(\overline{\eta_a}, o)\big| + 1 + \big|(a\eta_a, y)\big|.$$

In the last equation we splitted off the necessary walking step from o to a and 'shifted' (η_a, ay) isometrically by multiplying from the left with $(\mathbf{0}, a)$. Observe that $\big|(a\eta_a, y)\big|$ equals the minimal distance of a walk starting in a, then realizing the configuration η_a before finally reaching ay. Note also that $a\overline{C_a} = C_a$ and $aC_a = \overline{C_a}$. See Figure 7.1.

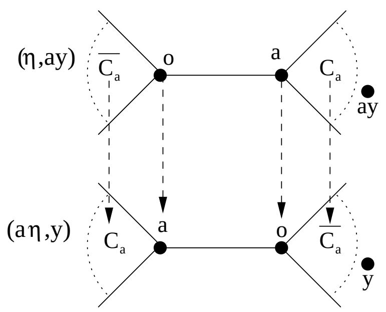


Figure 7.1: Shift by $(\mathbf{0}, a)$ from (η, ay) to $(a\eta, y)$

Let $\eta' := a\eta$. Then $(\mathbf{0}, a)(\eta, x) = (\eta', y)$. Furthermore, $\eta'_a = a\overline{\eta_a}$ and $\overline{\eta'_a} = a\eta_a$. Hence,

$$\big|(\eta', y)\big| = \big|(\eta'_a, o)\big| + \big|(\overline{\eta'_a}, y)\big| = \big|(\eta'_a, o)\big| + \big|(a\eta_a, y)\big|.$$

As $\overline{\eta_a}(w) = 1$ if and only if $\eta'_a(aw) = 1$, it follows that

$$\big|(\eta'_a, o)\big| = \begin{cases} 2 + \big|(\overline{\eta_a}, o)\big|, & \text{if } \overline{\eta_a} \not\equiv \mathbf{0} \\ 0, & \text{if } \overline{\eta_a} \equiv \mathbf{0} \end{cases}.$$

This finishes the proof. $\square$

Analogously:

Proposition 7.5. *Let $a \in \mathcal{S}$, $x \in \overline{C_a}$ and $\eta \in \mathcal{N}$. Then*

$$\big|(\mathbf{0}, a)(\eta, x)\big| - \big|(\eta, x)\big| = \begin{cases} -1, & \text{if } \eta_a \not\equiv \mathbf{0} \\ 1, & \text{if } \eta_a \equiv \mathbf{0} \end{cases}.$$

Proof. Observe again that $\eta_a(w) = 1$, $\overline{\eta_a}(w) = 1$ respectively, if and only if $(a\eta_a)(aw) = 1$, $(a\overline{\eta_a})(aw) = 1$ respectively, for any $w \in \mathcal{T}_q$. We obtain

$$\big|(\eta, x)\big| = \big|(\eta_a, o)\big| + \big|(\overline{\eta_a}, x)\big|.$$

Furthermore,

$$\big|(\eta_a, o)\big| = \begin{cases} 2 + \big|(a\eta_a, o)\big|, & \text{if } \eta_a \not\equiv \mathbf{0} \\ 0, & \text{if } \eta_a \equiv \mathbf{0} \end{cases}.$$

Let $\eta' := a\eta$. Then $(\mathbf{0}, a)(\eta, x) = (\eta', ax)$. Furthermore, $\eta'_a = a\overline{\eta_a}$ and $\overline{\eta'_a} = a\eta_a$. See Figure 7.2.

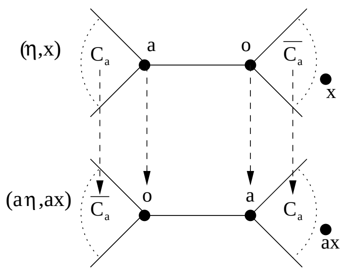


Figure 7.2: Shift by $(\mathbf{0}, a)$ from (η, x) to $(a\eta, ax)$ with $x \in \overline{C_a}$

Hence,

$$\big|(\eta', ax)\big| = \big|(\overline{\eta'_a}, o)\big| + \big|(\eta'_a, ax)\big| = \big|(a\eta_a, o)\big| + 1 + \big|(\overline{\eta_a}, x)\big|.$$

This finishes the proof. □

Proposition 7.6. *Let $(\eta, x) \in \mathcal{N} \times \mathcal{T}_q$. Then*

$$\big|(\mathbb{1}_o, o)(\eta, x)\big| - \big|(\eta, x)\big| = \begin{cases} 1, & \text{if } \eta(o) = 0 \\ -1, & \text{if } \eta(o) = 1 \end{cases}.$$

Proof. Obviously, $(\mathbb{1}_o, o)(\eta, x)$ and (η, x) differ only by the lamp state at the root o, as $(\mathbb{1}_o \oplus \eta)(o) = 1 - \eta(o)$. This proves the claim. □

Propositions 7.4, 7.5 and 7.6 show that $Y_{g,n} \in \{-1, 1\}$. More precisely, $Y_{g,n}$ remains unchanged after the last visit in o, that is, $Y_{g,n}$ converges almost surely. By Lebesgue's Dominated Convergence Theorem, convergence of $(\mathbb{E}[|Z_{n+1}| - \mathbb{E}[|Z_n|])_{n\in\mathbb{N}}$ follows. Now we want to compute the integrals $\int Y_{g,n}\, d\mathbb{P}$. For this purpose, we need the following probabilities:

Lemma 7.7.

$$\mathbb{P}\big[\eta_\infty(o) = 0\big] = \frac{q - 2 + p}{(1+p)\,q - 2} \quad \textit{and} \quad \mathbb{P}\big[\eta_\infty(o) = 1\big] = \frac{p(q-1)}{(1+p)\,q - 2}.$$

Proof. Let

$$\begin{aligned}
\widetilde{U} &:= \mathbb{P}\big[T_o < \infty, X_1 \neq o\big] = \sum_{a\in\mathcal{S}} \mu\big((\mathbf{0},a)\big)F = \frac{1-p}{q-1},\\
\widetilde{G} &:= \sum_{n\geq 0} \mathbb{P}\big[X_n = o, \forall j<n : \neg\big(X_j = o \wedge \mathbf{i}_{j+1} = (\mathbb{1}_o, o)\big)\big]\\
&= \frac{1}{1-\widetilde{U}} = \frac{q-1}{q-2+p}.
\end{aligned}$$

Now we can compute the proposed probabilities:

$$\begin{aligned}
\mathbb{P}\big[\eta_\infty(o) = 0\big] &= \sum_{m\geq 0} \big(\widetilde{G}\cdot p\big)^{2m} \cdot \widetilde{G}\cdot(1-p)\cdot(1-F)\\
&= \frac{q-2+p}{(1+p)\,q-2},\\
\mathbb{P}\big[\eta_\infty(o) = 1\big] &= 1 - \mathbb{P}\big[\eta_\infty(o) = 0\big] = \frac{p(q-1)}{(1+p)\,q-2}.
\end{aligned}$$

□

By Propositions 7.4, 7.5, 7.6 and Lemma 7.7, we obtain for $g = (\mathbb{1}_o, o)$

$$\int Y_{g,n}\, d\mathbb{P} = \mathbb{P}\big[\eta_\infty(o) = 0\big] - \mathbb{P}\big[\eta_\infty(o) = 1\big] = \frac{(1-p)(q-2)}{(1+p)\,q-2}$$

and for $g = (\mathbf{0}, a)$ with $a \in \mathcal{S}$

$$\begin{aligned}
\int Y_{g,n}\, d\mathbb{P} &= \mathbb{P}\big[X_\infty \text{ has first letter } a,\ \eta_\infty(\overline{C_a}) \not\equiv \mathbf{0}\big]\\
&\quad -\mathbb{P}\big[X_\infty \text{ has first letter } a,\ \eta_\infty(\overline{C_a}) \equiv \mathbf{0}\big]\\
&\quad +\mathbb{P}\big[X_\infty \text{ does not have first letter } a,\ \eta_\infty(C_a) \equiv \mathbf{0}\big]\\
&\quad -\mathbb{P}\big[X_\infty \text{ does not have first letter } a,\ \eta_\infty(C_a) \not\equiv \mathbf{0}\big]\\
&= \Big(\frac{1}{q} - \nu_2\Big) - \nu_2 + \Big(\frac{q-1}{q} - \nu_1\Big) - \nu_1\\
&= 1 - 2\nu_1 - 2\nu_2.
\end{aligned}$$

Now we can give two explicit formulas for the rate of escape:

Theorem 7.8.

$$\begin{aligned}
\ell_{\mathrm{WoS}} &= \frac{(1-p)(q-2)}{q} \cdot \Big(1 + 2q\nu_1 + \frac{pq}{(1+p)\,q-2}\Big)\\
&= \frac{(1-p)(q-2)}{q} \cdot \Big(\frac{q+1}{q-1} - \frac{2q}{q-1}\nu_2 + \frac{pq}{(1+p)\,q-2}\Big)
\end{aligned}$$

Proof. By Lebesgue's Dominated Convergence Theorem and the above computations, we get

$$\begin{aligned}
\ell_{\mathrm{WoS}} &= \sum_{g\in\mathcal{S}_{\mathcal{L}_q}} \mu(g) \int \lim_{n\to\infty} \big(\ell(gZ_n)-\ell(Z_n)\big)\, d\mathbb{P} \\
&= \sum_{a\in\mathcal{S}} \Big(\mu\big((\mathbf{0},a)\big)\cdot\big(1-2\nu_1-2\nu_2\big)\Big) + \mu\big((\mathbb{1}_o,o)\big)\cdot \frac{(1-p)(q-2)}{(1+p)\,q-2} \\
&= (1-p)\cdot(1-2\nu_1-2\nu_2)+\frac{p(1-p)(q-2)}{(1+p)\,q-2}.
\end{aligned}$$

The rest follows by substituting $\nu_1 = \frac{1}{q-1}(\frac{1}{q}-\nu_2)$, $\nu_2 = \frac{1}{q}-(q-1)\nu_1$ respectively. □

Remark: Observe that $\nu_2 = \check{G}\cdot(1-p)/q\cdot(1-F)$ holds, where

$$\check{G} = \sum_{\eta\in\mathcal{N}'} G(\eta) \quad \text{with} \quad G(\eta) = \sum_{n\geq 0} p^{(n)}\big((\mathbf{0},o),(\eta,o)\big)$$

for $\mathcal{N}' := \{\eta\in\mathcal{N} \mid \forall w\in\overline{C_{a_1}} : \eta(w)=0\}$. The functions $G(\eta)$ are Green functions evaluated at 1. As Green functions are in general hard to compute or even often not computable and since the structure of the Cayley graph of $\mathcal{L}_q$ is very complex, we are only able to give a lower and upper bound for ℓ_{WoS} by estimating ν_1 and ν_2 from below. For this purpose, we need the following lemma:

Lemma 7.9. *Let $z=(\eta_x,x)\in\mathcal{L}_q$ and let $y\in\mathcal{T}_q$ be a neighbour of x in the tree. Then the probability that the lamplighter, starting at x with configuration η, reaches y without changing any lamps during this walk is*

$$\bar{F} := \mathbb{P}_z\big[T_y<\infty, \forall k<T_y : \mathbf{i}_k \neq (\mathbb{1}_o,o)\big] = \frac{q-\sqrt{q^2-4(q-1)(1-p)^2}}{2(q-1)(1-p)}.$$

Proof. By vertex-transitivity, we get the recursive equation

$$\bar{F} = \mu\big((\mathbf{0},a_i)\big) + \sum_{a_j\in\mathcal{S}\setminus\{a_i\}} \mu\big((\mathbf{0},a_j)\big)\bar{F}^2 \quad \text{for any } a_i\in\mathcal{S}$$

with solutions

$$\bar{F} = \frac{q\pm\sqrt{q^2-4(q-1)(1-p)^2}}{2(q-1)(1-p)},$$

where the right one has to to fulfill $\bar{F}<1$. This proves the lemma. □

Now we can estimate ν_1 and ν_2 from below:

Lemma 7.10.

$$\begin{aligned} \nu_1 &\geq \frac{p}{(1+p)\,q^2-2q} =: \widehat{\nu}_1 \quad \textit{and} \\ \nu_2 &\geq \frac{\widehat{G}^2}{1-\widehat{G}^2p^2}\frac{(1-p)(q-2)}{q(q-1)} =: \widehat{\nu}_2, \end{aligned}$$

where

$$\widehat{G} = \frac{2(q-1)}{q-2+\sqrt{q^2-4(q-1)(1-p)^2}}.$$

Proof. We restrict the event $\big[\eta_\infty(C_{a_1}) \not\equiv \mathbf{0}\big]$ to the event $\big[\eta_\infty(a_1) = 1\big]$. Hence,

$$\begin{aligned} \nu_1 &\geq F \cdot \sum_{m\geq 0} \big(\widetilde{G}\cdot p\big)^{2m+1} \cdot \widetilde{G} \cdot \frac{1-p}{q} \cdot \big(1-F\big) \\ &= \frac{p}{(1+p)\,q^2-2q}. \end{aligned}$$

For the computation of the lower bound of ν_2, we introduce some further notation:

$$\begin{aligned} \widehat{U} &:= \mathbb{P}\big[T_o<\infty, \forall j<T_o : \neg\big(X_j \in \overline{C_{a_1}} \wedge \mathbf{i}_{j+1} = (\mathbb{1}_o, o)\big)\big] \\ &= \frac{q-1}{q}(1-p)\cdot \bar{F} + \frac{1-p}{q}\cdot F, \\ \widehat{G} &:= \sum_{n\geq 0} \mathbb{P}\big[X_n = o, \forall j<n : \neg\big(X_j \in \overline{C_{a_1}} \wedge \mathbf{i}_{j+1} = (\mathbb{1}_o, o)\big)\big] \\ &= \frac{1}{1-\widehat{U}}. \end{aligned}$$

We restrict the event $\big[\eta_\infty(\overline{C_{a_1}}) \equiv \mathbf{0}\big]$ to the event that no lamps in $\overline{C_{a_1}} \setminus \{o\}$ are switched on, that is, $\eta_n(\overline{C_{a_1}} \setminus \{o\}) \equiv \mathbf{0}$ for all $n \in \mathbb{N}$, while we allow to switch the lamp at o for an even number of switches. This yields

$$\begin{aligned} \nu_2 &\geq \sum_{m\geq 0} \big(\widehat{G}\cdot p\big)^{2m} \cdot \widehat{G} \cdot \frac{1-p}{q} \cdot (1-F) \\ &= \frac{\widehat{G}}{1-\widehat{G}^2p^2} \cdot \frac{(1-p)(q-2)}{q(q-1)}. \end{aligned}$$

□

Now we can give an upper and lower bound for the rate of escape:

Corollary 7.11.

$$\begin{aligned}\ell_{\mathrm{WoS}} &\geq \frac{(1-p)(q-2)}{q}\cdot\frac{q-2+2p(q+1)}{(1+p)\,q-2} =: \ell_{\mathrm{low}} \qquad \textit{and}\\ \ell_{\mathrm{WoS}} &\leq \frac{(1-p)(q-2)}{q}\cdot\Big(\frac{q+1}{q-1}-\frac{2q}{q-1}\hat{\nu}_2+\frac{pq}{(1+p)\,q-2}\Big) =: \ell_{\mathrm{up}}\end{aligned}$$

□

Observe that the lower bound satisfies

$$\ell_{\mathrm{low}} > \frac{(1-p)(q-2)}{q} = \lim_{n\to\infty}\frac{d(o,X_n)}{n}.$$

Numerical sample computations are presented at the end of the next section. The technique used in Section 6.1 provides also

$$\ell_{\mathrm{WoS}} \geq \frac{(1-p)(q-2)}{q}\cdot\Big(1+p\cdot(1-p)\cdot\frac{q-2}{q-1}\Big).$$

However, it can be shown that ℓ_{low} is stricty bigger than this lower bound.

7.3 Another Lower Bound

We construct another lower bound for ℓ_{WoS}, which is better than ℓ_{low} if $p \leq (q-2)/(q-1)$. For this purpose, we give another lower bound for ν_1 and then apply Theorem 7.8.

Observe that

$$\nu_1 = F\cdot\underbrace{\mathbb{P}\big[a \text{ is first letter of } X_\infty, \eta_\infty(\overline{C_{a_1}}) \not\equiv \mathbf{0}\big]}_{=:\nu_3}.$$

Observe that $\eta_\infty(\overline{C_{a_1}}) \not\equiv \mathbf{0}$ means that at least one lamp in $\overline{C_{a_1}}$ rests on forever. Now we distinguish which of the lamps in $\overline{C_{a_1}} \cap \operatorname{supp}\eta_\infty$ is the first lamp to be switched on, while it is allowed to turn it off temporarily. More formally, define the random variable $\mathbf{l}_1$ such that $\mathbf{l}_1 = x \in \overline{C_{a_1}} \cap \operatorname{supp}\eta_\infty$ if $X_n = X_{n+1} = x$ holds for some $n \in \mathbb{N}_0$ with $\eta_m(y) = 0$ for all $m < n$ and all $y \in \overline{C_{a_1}} \cap \operatorname{supp}\eta_\infty$. It is sufficient to define $\mathbf{l}_1$ only on the event $[\eta_\infty(\overline{C_{a_1}}) \not\equiv \mathbf{0}]$. Define also

$$\begin{aligned} L &:= \sum_{n\geq 1}\mathbb{P}\big[X_n = a_1, \forall m\in\{1,\dots,n\}: X_m \neq o\big]\\ &= \frac{1-p}{q}\cdot\sum_{n\geq 0}\Big(\frac{q-1}{q}(1-p)F+p\Big)^n = \frac{1}{q-1}\end{aligned}$$

and

$$\begin{aligned}\bar{G} &:= \sum_{n\ge 0} \mathbb{P}\big[X_n = o, \forall k \le n : \mathbf{i}_k \neq (\mathbb{1}_o, o)\big] \\ &= \sum_{n\ge 0} \big((1-p)\bar{F}\big)^n = \frac{1}{1-(1-p)\bar{F}}.\end{aligned}$$

Now

$$\begin{aligned}\nu_3 &= \sum_{x\in\overline{C_{a_1}}} \mathbb{P}[a_1 \text{ is first letter of } X_\infty, \eta_\infty(\overline{C_{a_1}}) \not\equiv \mathbf{0}, \mathbf{l}_1 = x] \\ &\ge \sum_{x\in\overline{C_{a_1}}} \bar{F}^{d(o,x)} \cdot \bar{G} \cdot \sum_{m\ge 0} \big(p\,\widetilde{G}\big)^{2m+1} \cdot L^{d(o,x)} \cdot \frac{1-p}{q} \cdot (1-F) \\ &= \frac{\bar{G}\widetilde{G}p}{1-p^2\widetilde{G}^2} \cdot \frac{1-p}{q} \cdot \frac{q-2}{q-1} \cdot \sum_{n\ge 0} (q-1)^n \big(\bar{F}\cdot L\big)^n \\ &= \frac{\bar{G}\widetilde{G}p}{1-p^2\widetilde{G}^2} \cdot \frac{1-p}{q} \cdot \frac{q-2}{q-1} \cdot \frac{1}{1-\bar{\bar{F}}} \\ &= \frac{p(q-2+p)}{q\big((1+p)\,q-2\big)\big(1-\bar{F}\big)\big(1-(1-p)\bar{F}\big)} =: \widehat{\nu_3}.\end{aligned}$$

Thus:

Corollary 7.12.

$$\ell_{\mathrm{WoS}} \ge \frac{(1-p)(q-2)}{q} \cdot \Big(1 + \frac{2q}{q-1}\widehat{\nu_3} + \frac{pq}{(1+p)\,q-2}\Big) = \ell_{\mathrm{low},2}$$

□

With the help of MATHEMATICA we can show that $\ell_{\mathrm{low},2} \ge \ell_{\mathrm{low}}$ if and only if $p \le (q-2)/(q-1)$.

Table 7.3 compares the values of the trivial lower bound given by Lemma 7.1, namely $\lim_{n\to\infty} d(o, X_n)/n = (1-p)(q-2)/q$, the lower bounds ℓ_{low} and $\ell_{\mathrm{low},2}$ and the upper bound ℓ_{up} for different values of q and p. The *relative precision of the approximation* is the quotient

$$\frac{\ell_{\mathrm{up}} - \max\{\ell_{\mathrm{low}}, \ell_{\mathrm{low},2}\}}{1 - \lim_{n\to\infty} d(o, X_n)/n},$$

which decreases when the degree q of the tree increases: large q yields tighter bounds.

q	p	$\lim_{n\to\infty} \frac{d(o,X_n)}{n}$	ℓ_{low}	$\ell_{\text{low},2}$	ℓ_{up}	relative precision
3	4/5	0.067	0.145098	0.144410	0.157358	0.01314
3	2/3	0.111	0.234567	0.233467	0.253779	0.02161
3	1/2	0.167	0.333	0.333	0.359733	0.03167
3	1/4	0.25	0.428571	0.438050	0.461289	0.03168
5	4/5	0.12	0.216	0.215942	0.221533	0.00629
5	2/3	0.2	0.347368	0.347629	0.355735	0.01013
5	1/2	0.3	0.490909	0.492585	0.501825	0.01320
5	1/4	0.45	0.635294	0.641344	0.647154	0.01056
10	4/5	0.16	0.256	0.256029	0.257516	0.00177
10	2/3	0.267	0.412121	0.412311	0.414351	0.00278
10	1/2	0.4	0.584615	0.585277	0.587408	0.00355
10	1/4	0.6	0.771429	0.773099	0.774203	0.00276
20	4/5	0.18	0.273176	0.273189	0.273569	0.00046
20	2/3	0.3	0.440426	0.440487	0.440995	0.00073
20	1/2	0.45	0.626785	0.626975	0.627483	0.00093
20	1/4	0.675	0.836413	0.836835	0.837079	0.00075

Figure 7.3: Sample computations of lower and upper bounds

7.4 Switch-Walk-Switch Random Walk

We now investigate the associated Switch-Walk-Switch lamplighter random walk on the homogeneous tree $\mathcal{T}_q$ and its drift in the context of Section 6.4.1. That is, we have $p_0(x,y) = 1/q$ for any pairs of neighbours $x, y \in \mathcal{T}_q$ and for $p \in (0,1)$ we define $\mu_x(\mathbb{1}_x) = p$ and $\mu_x(\mathbf{0}) = 1-p$. Each edge of $\mathcal{T}_q$ shall have weight 1 and additionally we set $\delta_{\mathcal{L}} = 0$. Furthermore, $\mathcal{T}_q \setminus B(x,0)$ has $q \geq 3$ different connected components, that is, $r = 0$.

Then $|z|$ is the minimal weight of all paths in $\mathbb{Z}_2 \wr \mathcal{T}_q$ joining $(\mathbf{0}, o)$ with $z \in \mathcal{N} \times \mathcal{T}_q$, while $d(\cdot,\cdot)$ is again the natural graph metric on $\mathcal{T}_q$. In other words, $|z|$ is the distance of o to z in the Caley graph of the wreath product $\mathcal{L}_q$ w.r.t. the set of generators

$$\mathcal{S}^*_{\mathcal{L}_q} := \big\{(\mathbb{1}_A, a) \mid a \in \mathcal{S}, A \in \{\emptyset, \{o\}, \{a\}, \{o,a\}\}\big\}.$$

In this case it is well-known that $\lim_{n\to\infty} d(o,X_n)/n = (q-2)/q$. Our aim is to give a tighter lower bound for $\ell_{\mathrm{SWS}} := \lim_{n\to\infty} |Z_n|/n$ than given by

$$\frac{q-2}{q} \cdot \frac{q^3(q-1)+4(q-2)p(1-p)}{q^3(q-1)}, \tag{7.2}$$

which is extrapolated from relation (6.11).

We proceed similarly to Section 6.2.1, but with some modifications. As $r=0$ and the metric $d(\cdot,\cdot)$ is integer-valued, ψ becomes the identity on $\mathbb{N}_0$ and

$$\mathbf{e}_k \quad := \quad \min\big\{m \in \mathbb{N}_0 \;\big|\; d(o,X_m) = k \wedge \forall n \geq m : X_n \in C_{X_m}\big\}$$

for $k \in \mathbb{N}_0$. For $k \in \mathbb{N}$, the pseudo-increments now become

$$\Delta_k := \begin{cases} 0, & \text{if } \eta_{\mathbf{e}_k}(w) = 0 \text{ for all } w \in C_{X_{\mathbf{e}_{k-1}}} \setminus (C_{X_{\mathbf{e}_k}} \cup \{X_{\mathbf{e}_{k-1}}\}) \\ 2, & \text{otherwise} \end{cases}. \tag{7.3}$$

The set $C_{X_{\mathbf{e}_{k-1}}} \setminus (C_{X_{\mathbf{e}_k}} \cup \{X_{\mathbf{e}_{k-1}}\})$ is the union of the cones C_z, where z is a forward neighbour of $X_{\mathbf{e}_{k-1}}$ distinct from $X_{\mathbf{e}_k}$. The pseudo-increment Δ_k represents a lower bound for the length of a possible deviation inside $C_{X_{\mathbf{e}_{k-1}}} \setminus C_{X_{\mathbf{e}_k}}$, when walking from o to X_n, where $\mathbf{e}_k < n$, with restoring the configuration η_n. Note that a shortest tour from o to X_n does not visit the set $C_{X_{\mathbf{e}_{k-1}}} \setminus (C_{X_{\mathbf{e}_k}} \cup \{X_{\mathbf{e}_{k-1}}\})$. If at time $\mathbf{e}_{k-1}$ the lamplighter stands at $g = g'a_1 \in \mathcal{T}_q$, then walks to ga_i, $i \notin \{1,q\}$, thereby switching the lamp at ga_i on, walks back to g without flipping the lamp state at ga_i, followed by walking to ga_q and rests henceforth in C_{ga_q}, then $\Delta_k = 2$. See Figure 7.4. Observe again that this setting constitutes only a special case of Section 6.2.1.

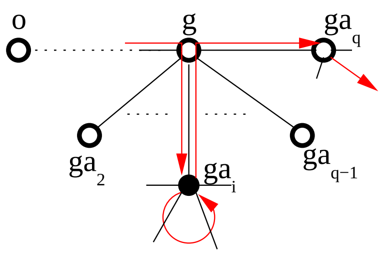


Figure 7.4: Interpretation of Δ_k

Analogously, we have for all $k \geq 1$

$$\big|(\eta_{\mathbf{e}_k}, Z_{\mathbf{e}_k})\big| \geq k + \sum_{j=1}^{k} \Delta_j. \tag{7.4}$$

To estimate the distribution of Δ_k, we distinguish if lamps are on in the set $C_{X_{\mathbf{e}_{k-1}}} \setminus \{X_{\mathbf{e}_{k-1}}\}$ at time $\mathbf{e}_{k-1}$ or not and if lamps are on in the set $C_{X_{\mathbf{e}_{k-1}}} \setminus \big(C_{X_{\mathbf{e}_k}} \cup \{X_{\mathbf{e}_{k-1}}\}\big)$ at time $\mathbf{e}_k$. For $x \in \mathcal{T}_q \setminus \{o\}$, we use the notation x^- to express the unique neighbour of x closer to o. For $k \in \mathbb{N}$, let

$$\begin{aligned}
E &:= \big\{(\eta, x) \in \mathcal{N} \times (\mathcal{T}_q \setminus \{o\}) \;\big|\; \exists w \in C_{x^-} \setminus (C_x \cup \{x^-\}) : \eta(w) = 1\big\},\\
E_{k,0} &:= \big\{(\eta, x) \in \mathcal{N} \times \mathcal{T}_q \;\big|\; |x| = k, \forall w \in C_x \setminus \{x\} : \eta(w) = 0\big\} \quad \text{and}\\
E_{k,2} &:= \big\{(\eta, x) \in \mathcal{N} \times \mathcal{T}_q \;\big|\; |x| = k, \exists w \in C_x \setminus \{x\} : \eta(w) = 1\big\}.
\end{aligned}$$

Observe that for $k \geq 2$ and $r \in \{0, 2\}$, it is

$$\mathbb{P}\big[Z_{\mathbf{e}_{k-1}} \in E_{k-1,r}\big] = \sum_{m \geq 0} \sum_{(\eta,x) \in E_{k-1,r}} \mathbb{P}\big[X_{m-1} = x^-, Z_m = (\eta, x)\big] \cdot \big(1 - F\big).$$

Thus,

$$\begin{aligned}
& \mathbb{P}[\Delta_k = 2 \mid Z_{\mathbf{e}_{k-1}} \in E_{k-1,r}]\\
= \; & \frac{1}{\mathbb{P}\big[Z_{\mathbf{e}_{k-1}} \in E_{k-1,r}\big]} \sum_{m \geq 0} \sum_{(\eta,x) \in E_{k-1,r}} \mathbb{P}\big[X_{m-1} = x^-, Z_m = (\eta, x)\big] \cdot\\
& \cdot \Big(\sum_{l \geq 1} \mathbb{P}_{(\eta,x)}\big[\forall \tau \leq l : X_\tau \neq x^-, X_{l-1} = x, (\eta_l, X_l) \in E\big] \Big) \cdot \big(1 - F\big)\\
\geq \; & \inf_{(\eta,x) \in E_{k-1,r}} \sum_{l \geq 1} \mathbb{P}_{(\eta,x)}\big[\forall \tau \leq l : X_\tau \neq x^-, X_{l-1} = x, (\eta_l, X_l) \in E\big].
\end{aligned}$$

Now we can prove:

Lemma 7.13. *We have $\mathbb{E}[\Delta_k] \geq B$ for all $k \in \mathbb{N}$, where*

$$B := \frac{4}{q^3} \cdot (q-1) \cdot (q-2) \cdot p \cdot (1-p) > 0.$$

Proof. Let $k \in \mathbb{N}$. Walking to a forward neighbour of $Z_{\mathbf{e}_{k-1}}$ with switching the lamp there, walking back to $Z_{\mathbf{e}_{k-1}}$, followed by a walk to some other forward neighbour of $Z_{\mathbf{e}_{k-1}}$, we get by the above computations:

$$\mathbb{P}[\Delta_k = 2 \mid Z_{\mathbf{e}_{k-1}} \in E_{k-1,0}] \geq 2 \cdot (q-1) \cdot \frac{p(1-p)}{q^2} \cdot \frac{q-2}{q} = \frac{1}{2} B > 0.$$

If $Z_{\mathbf{e}_{k-1}} \in E_{k-1,2}$, then there is a forward subcone C_w of $C_{Z_{\mathbf{e}_{k-1}}}$ such that at least one lamp in C_w is on. With a walking step from $Z_{\mathbf{e}_{k-1}}$ into one of the forward subcones of $C_{Z_{\mathbf{e}_{k-1}}}$ different from C_w, we also obtain

$$\mathbb{P}[\Delta_k = 2 \mid Z_{\mathbf{e}_{k-1}} \in E_{k-1,2}] \geq \frac{q-2}{q} \geq \frac{1}{2} B.$$

Thus, we obtain for $k \geq 2$

$$\begin{aligned} \mathbb{E}[\Delta_k] &= \mathbb{P}[Z_{\mathbf{e}_{k-1}} \in E_{k-1,0}] \cdot \mathbb{E}[\Delta_k \mid Z_{\mathbf{e}_{k-1}} \in E_{k-1,0}] \\ &\quad + \mathbb{P}[Z_{\mathbf{e}_{k-1}} \in E_{k-1,2}] \cdot \mathbb{E}[\Delta_k \mid Z_{\mathbf{e}_{k-1}} \in E_{k-1,2}] \geq B > 0. \end{aligned}$$

We have to handle the case $k = 1$ separately: here, we have $\mathbb{P}[Z_{\mathbf{e}_0} \in E_{0,0}] = 1$ and thus

$$\mathbb{E}[\Delta_1] \geq 4 \cdot q \cdot \frac{p(1-p)}{q^2} \cdot \frac{q-1}{q} \geq B.$$

□

Now we can give another lower bound for the drift of the Switch-Walk-Switch lamplighter random walk on $\mathcal{T}_q$:

Theorem 7.14. *For the Switch-Walk-Switch lamplighter random walk on the homogeneous tree $\mathcal{T}_q$ with simple base random walk, assuming $\delta_{\mathcal{L}} = 0$,*

$$\ell_{\mathrm{SWS}} \geq \frac{q-2}{q} \cdot \frac{q^3 + 4(q-1)(q-2)p(1-p)}{q^3}.$$

Proof. The proof works analogously to the proof of Theorem 6.9, except that we use Δ_k as defined by (7.3), providing $\mathbb{E}[\Delta_k] \geq B$. □

Observe that the lower bound for ℓ_{SWS} given by the last theorem is considerably bigger than the lower bound given by (7.2).

It is also possible to construct lower and upper bounds for the rate of escape of this random walk by the technique used in Section 7.2. Besides the much greater complexity in the SWS-case, numerical computations show that those bounds are less tight than in the WoS-case of Section 7.2, that is, the spread between the lower and upper bound is bigger.

Acknowledgements

The author is at most grateful to Wolfgang Woess for numerous discussions on several problems and his help during the preparation of this dissertation. The author is also grateful to Donald Cartwright for several hints regarding content and exposition. Acknowledgements go also to the colleagues of the 'Institut für mathematische Strukturtheorie' at Graz University of Technology for several discussions on mathematical problems.

Bibliography

[1] D. Bertacchi. Random walks on Diestel-Leader graphs. *Abh. Math. Sem. Univ. Hamburg*, 71:205–224, 2001.

[2] S. Blachère and S. Brofferio. Internal diffusion limited aggregation on discrete groups having exponential growth. *Probab. Th. Rel. Fields*, in print.

[3] S. Blachère, P. Haïssinsky, and P. Mathieu. Asymptotic entropy and Green speed for random walks on groups. *Preprint*, 2006.

[4] P. Brémaud. *Markov Chains. Gibbs Fields, Monte Carlo Simulation, and Queues*. Springer, 1999.

[5] D. Cartwright, V. Kaimanovich, and W. Woess. Random walks on the affine group of local fields and of homogeneous trees. *Ann. Inst. Fourier (Grenoble)*, 44:1243–1288, 1994.

[6] D. Cartwright and P. Soardi. Random walks on free products, quotients, and amalgams. *Nagoya Math. J.*, 102:163–180, 1986.

[7] D. Cartwright and W. Woess. Isotropic random walks in a building of type $\tilde{A}_d$. *Mathematische Zeitschrift*, 247:101–135, 2004.

[8] Y. Derriennic. Quelques applications du théorème ergodique sous-additif. *Astérisque*, 74:183–201, 1980.

[9] A. Dyubina. Characteristics of random walks on wreath products of groups. *J. of Math. Sciences*, 107(5):4166–4171, 2001.

[10] A. Erschler. On drift and entropy growth for random walks on groups. *Ann. of Probab.*, 31(3):1193–1204, 2003.

[11] A. Erschler. On the asymptotics of drift. *J. of Math. Sciences*, 121(3):2437–2440, 2004.

[12] H. Furstenberg. Non commuting random products. *Trans. Amer. Math. Soc.*, 108:377–428, 1963.

[13] Y. Guivarc'h. Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire. *Astérisque*, 74:47–98, 1980.

[14] V. Kaimanovich. Poisson boundary of discrete groups. *Preprint.*

[15] V. Kaimanovich and A. Vershik. Random walks on discrete groups: boundary and entropy. *Ann. of Probab.*, 11:457–490, 1983.

[16] V. Kaimanovich and W. Woess. The Dirichlet problem at infinity for random walks on graphs with a strong isoperimetric inequality. *Probab. Theory Related Fields*, 91:445–466, 1992.

[17] V. Kaimanovich and W. Woess. Boundary and entropy of space homogeneous Markov chains. *Ann. of Probab.*, 30:323–363, 2002.

[18] A. Karlsson and F. Ledrappier. On laws of large numbers for random walks. *Ann. of Probab.*, Preprint.

[19] H. Kesten. *Symmetric Random Walks on Groups.* PhD thesis, Graduate School of Cornell University, 1957.

[20] J. Kingman. The ergodic theory of subadditive processes. *J. Royal Stat. Soc., Ser. B*, 30:499–510, 1968.

[21] F. Ledrappier. Some asymptotic properties of random walks on free groups. In *CRM Proceedings and Lecture Notes*, volume 28, pages 117–152. CRM, 2001.

[22] R. Lyndon and P. Schupp. *Combinatorial Group Theory.* Springer-Verlag, 1977.

[23] R. Lyons, R. Pemantle, and Y. Peres. Random walks on the lamplighter group. *Ann. of Probab.*, 24(4):1993–2006, 1996.

[24] J. Mairesse. Random walks on groups and monoids with a Markovian harmonic measure. *Research Report LIAFA 2004-005, Univ. Paris 7*, 2004.

[25] J. Mairesse. Randomly growing braid on three strands and the manta ray. *Report LIAFA 2005-001, Univ. Paris 7*, 2005.

[26] J. Mairesse and F. Mathéus. Random walks on free products of cyclic groups and on artin groups with two generators. *Research Report LIAFA 2004-006, Univ. Paris 7*, 2004.

[27] P. Mathieu. Carne-Varopoulos bounds for centered random walks. *Ann. of Probab.*, 34(3):987–1011, 2006.

[28] J. McLaughlin. *Random Walks and Convolution Operators on Free Products.* PhD thesis, New York Univ., 1986.

[29] B. Mohar. Some relations between analytic and geometric properties of infinite graphs. *Discrete Math.*, 95(193–219), 1991.

[30] T. Nagnibeda and W. Woess. Random walks on trees with finitely many cone types. *J. Theoret. Probab.*, 15:399–438, 2002.

[31] J. Parkinson. Isotropic random walks on affine buildings. *Preprint*, 2005.

[32] A. Paterson. *Amenability.* Number 29 in Mathematical Surveys and Monographs Series. American Mathematical Society, 1988.

[33] C. Pittet. On the isoperimetry of graphs with many ends. *Colloq. Math.*, 78:307–318, 1998.

[34] D. Revelle. Rate of escape of random walks on wreath products and related groups. *Ann. of Probab.*, 31(4):1917–1934, 2003.

[35] M. Salvatori. Random walks on generalized lattices. *Monatshefte für Mathematik*, 121:145–161, 1996.

[36] S. Sawyer. Isotropic random walks on a tree. *Zeitschrift f. Wahrscheinlichkeitstheorie*, Verw. Geb. 42:279–292, 1978.

[37] S. Sawyer and T. Steger. The rate of escape for anisotropic random walks in a tree. *Probab. Theory Related Fields*, 76:207–230, 1987.

[38] N. T. Varopoulos. Long range estimates for Markov chains. *Bull. Sc. math.*, 109:225–252, 1985.

[39] D. Voiculescu. Addition of certain non-commuting random variables. *J. Funct. Anal.*, 66:323–346, 1986.

[40] W. Woess. A description of the Martin boundary for nearest neighbour random walks on free products. *Probability Measures on Groups*, VII:203–215, 1985.

[41] W. Woess. Nearest neighbour random walks on free products of discrete groups. *Boll. Un. Mat. Ital.*, 5-B:961–982, 1986.

[42] W. Woess. Boundaries of random walks on graphs and groups with infinitely many ends. *Israel Journal of Mathematics*, 68(3):271–301, 1989.

[43] W. Woess. *Random Walks on Infinite Graphs and Groups.* Cambridge University Press, 2000.

[44] W. Woess. A note on the norms of transition operators on lamplighter graphs and groups. *International Journal of Algebra and Computation*, Vol. 15(Nos. 5 & 6):1261–1272, 2005.